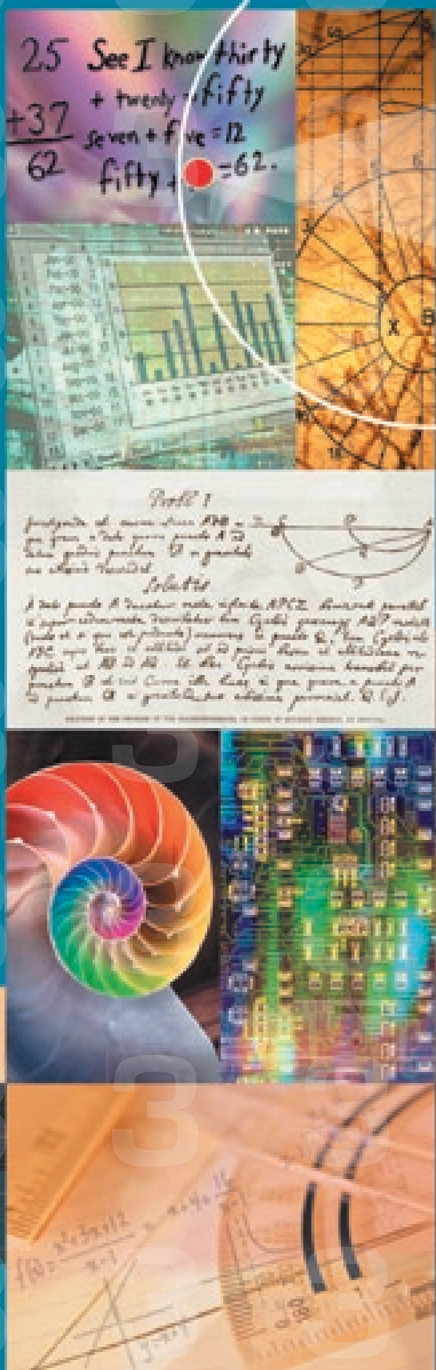
Focus in Grade 3

Teaching with Curriculum Focal Points



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Focus in Grade 3

Teaching with Curriculum Focal Points

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The Teaching with Curriculum Focal Points series consists of grade-level publications designed to support teachers, supervisors, and coordinators as they begin the discussion of a more focused curriculum across and within prekindergarten through grade 8, as presented in *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics*.

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Focus in Grade 3

Teaching with Curriculum Focal Points

3

Jane F. Schielack, *Series Advisor*
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Contents

Contents

Contents

PREFACE

On September 12, 2006, the National Council of Teachers of Mathematics released *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* to encourage discussions at the national, state, and district levels on the importance of designing a coherent elementary school mathematics curriculum focusing on the important mathematical ideas at each grade level. The natural question that followed the release of *Curriculum Focal Points* was "How do we translate this view of a focused curriculum into the classroom?"

Focus in Grade 3, one in a series of grade-level publications, is designed to support teachers, supervisors, and coordinators as they begin the discussion of a more focused curriculum across and within prekindergarten through grade 8, as presented in *Curriculum Focal Points. Focus in Grade 3*, in conjunction with the *Focus in Grade 4* and *Focus in Grade 5* books, will provide a strong foundation for mathematics in a focused curriculum across grades 3 through 5. In addition, teacher educators should find *Focus in Grade 3* useful as a vehicle for exploring with their preservice teachers the mathematical ideas and curriculum issues related to the suggested grade 3 Curriculum Focal Points.

The contributors to, and reviewers of, these publications, all active leaders in mathematics education and professional development, guided the creation of this grade-level book as a framework for lesson-study experiences in which teachers deepen their understanding of the mathematical ideas they will be teaching. This book describes and illustrates instructional progressions for the mathematical concepts and skills of each grade 3 Curriculum Focal Point, including powerful representational supports for teaching and learning that can facilitate understanding, stimulate productive discussions about mathematical thinking, and provide a foundation for fluency with the core ideas. Because these progressions cut across grades, you will see the same progressions for similar topics in grades 3, 4 and 5. Different parts of those progressions will appear as curriculum focal points in the different grades.

Whether you are working with your colleagues or individually, we hope you will find the discussions of the instructional progressions, representations, problems, and lines of reasoning valuable as you plan activities and discussions for your students and as you strive to help your students achieve the depth of understanding of important mathematical concepts necessary for their future success.

—Jane F. Schielack
Series Advisor

PREFACE TO *CURRICULUM FOCAL POINTS FOR PREKINDERGARTEN THROUGH GRADE 8 MATHEMATICS*

As states and local school districts implement more rigorous assessment and accountability systems, teachers often face long lists of mathematics topics or learning expectations to address at each grade level, with many topics repeating from year to year. Lacking clear, consistent priorities and focus, teachers stretch to find the time to present important mathematical topics effectively and in depth.

The National Council of Teachers of Mathematics (NCTM) is responding to this challenge by presenting *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.* Building on *Principles and Standards for School Mathematics* (NCTM 2000), this new publication is offered as a starting point in a dialogue on what is important at particular levels of instruction and as an initial step toward a more coherent, focused curriculum in this country.

The writing team for *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* consisted of nine members, with at least one university-level mathematics educator or mathematician and one pre-K–8 classroom practitioner from each of the three grade bands (pre-K–grade 2, grades 3–5, and grades 6–8). The writing team examined curricula from multiple states and countries as well as a wide array of researchers' and experts' writings in creating a set of focal points for pre-K–grade 8 mathematics.

On behalf of the Board of Directors, we thank everyone who helped make this publication possible.

Cathy Seeley
President, 2004–2006
National Council of Teachers of Mathematics

Francis (Skip) Fennell
President, 2006–2008
National Council of Teachers of Mathematics

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Jane F. Schielack
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1 Introduction

Purpose of This Guide

Your first question when looking at NCTM's Curriculum Focal Points might be, *How can I use NCTM's Focal Points with the local and state curriculum I am expected to teach?* NCTM's Curriculum Focal Points are not intended to be a national curriculum but have been developed to help bring more consistency to mathematics curricula across the country. Collectively, they constitute a framework of how a curriculum might be organized at each grade level, prekindergarten through grade 8. They are also intended to help bring about discussion within and across states and school districts about the important mathematical ideas to be taught at each grade level. Because of the current variation among states' curricula, the Curriculum Focal Points are not likely to match up perfectly with any one state's curriculum. This volume explores the mathematics emphasized at grade 3 in the focused curriculum suggested by the NCTM Curriculum Focal Points framework. Additional grade-level and grade-band books will be developed by NCTM to help teachers translate the Curriculum Focal Points identified for their grade level into coherent and meaningful instruction. Taken together, this grade 3 guide, along with the grade 4 and grade 5 guides and the grade 3–5 grade-band guide, can be used by groups of teachers in professional development experiences as well as by individual classroom teachers.

A curriculum is more than a collection of activities: It must be coherent, focused on important mathematics, and well articulated across the grades.

—The Curriculum Principle,
Principles and Standards for School Mathematics

Purpose of Curriculum Focal Points

The mathematics curriculum in the United States has often been characterized as a "mile wide and an inch deep." Many topics are studied each year—often reviewing much that was covered in previous years—and little depth is added each time the topic is addressed. In addition, because education has always been locally controlled in the United States, learning expectations can significantly differ by state and local school systems. NCTM's *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (2006) is the next step in helping states and local districts refocus their curricula. It provides an example of a focused and coherent curriculum in prekindergarten through grade 8 by identifying the most important mathematical topics, or "Focal Points," at each grade level. The Focal Points are not discrete topics to be taught and checked off, but rather a cluster of related knowledge, skills, and concepts. By organizing and prioritizing curriculum and instruction in prekindergarten–grade 8 around Focal Points at each grade level, teachers can foster more cumulative learning of mathematics by students, and students' work in the later grades will build on and deepen what they learned in the earlier grades. Organizing mathematics content in this way will help ensure a solid mathematical foundation for high school mathematics and beyond.

This guide provides an example of a focused and coherent curriculum in prekindergarten through grade 8 by identifying the most important mathematical topics, or "Focal Points," at each grade level.

Prior to the Curriculum Focal Points, the National Council of Teachers of Mathematics (NCTM) began the process of bringing about change to school mathematics programs in the 1980s, particularly with the first document to outline standards in mathematics, titled *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). That document provided major direction to states and school districts in developing their curricula. NCTM's *Principles and Standards for School Mathematics* (*Principles and Standards*) (2000) further elaborated on the ideas of the 1989 Standards, outlining learning expectations in the grade bands of prekindergarten–2, 3–5, 6–8, and 9–12. *Principles and Standards* also highlighted six principles, which include the Curriculum Principle, to offer guidance for developing mathematics programs. The Curriculum Principle emphasizes the need to link with, and build on, mathematical ideas as students progress through the grades, deepening their mathematical knowledge over time.

Impact of Focal Points on Curriculum, Instruction, and Assessment

Significant improvement can be made in the areas of curriculum, instruction, and assessment by identifying Focal Points at each grade level. At the curriculum level, Focal Points will allow for more rigorous and in-depth study of important mathematics at each grade level. This rigor will translate to a more meaningful curriculum that students can understand and apply. At the instructional level, Focal Points will allow teachers to more fully know the core topics they are responsible for teaching. Teachers will not be necessarily teaching *less* or *more* but will be able to teach *better*. Professional development can also be tailored to deepen teachers' knowledge of these Focal Points and connect these ideas in meaningful ways. Assessments can be designed that truly measure students' mastery of core topics rather than survey a broad range of disparate topics, thus allowing for closer monitoring of students' development. At the level of classroom assessment, having a smaller number of essential topics will help teachers have time to better determine what their students have learned and whether they have learned the material deeply enough to use and build on it in subsequent years. If state assessments are more focused as well, more detailed information can be gathered for districts and schools on areas for improvement.

Using This *Focus in Grade 3* Book

Many teachers tell us that they did not have an opportunity in their teacher-preparation programs to build sufficient understanding of some of the mathematical topics that they now teach. The discussion of the mathematical ideas presented here is detailed enough for teachers to begin building understanding of the mathematics contained in each grade 3 Focal Point. To further understand what mathematics students are expected to learn before grade 3 and in later grades, the teacher would benefit from examining the publications *Focus in Grade 2* and *Focus in Grade 4*. We suggest that teachers form study groups (such as those in lesson study or mathematics circles, or other learning communities) to read and discuss parts of this publication, to work together to build a deeper understanding of the mathematical topics in each Focal Point, and to plan how to help their students develop such understanding by adapting as needed their present grade 3 teaching and learning strategies and materials. A helpful approach for other teacher working groups has been to share student insights and questions and to look at student work to understand different ways that students are solving problems, to address errors, and to help students move forward in a progression that fosters both understanding and fluency. Because teachers' lives are busy and demanding, the reader is better served by concentrating on small portions of this publication at a time and working through them deeply instead of trying to do too much at once and getting discouraged. Teacher learning, like student learning, is a continual process that can be very rewarding.

Bringing Focus into the Classroom: Instruction That Builds Understanding and Fluency

Although the main goal of this publication is to present in more detail the mathematics in each of the Focal Points, some important pedagogical issues also need to be taken into account when creating an environment that supports focused instruction. Pedagogical principles for classrooms that do help students build understanding are outlined in *Principles and Standards for School Mathematics* (NCTM 2000) and in the National Research Council reports *Adding It Up* (Kilpatrick, Swafford, and Findell 2001) and *How Students Learn: Mathematics in the Classroom* (Donovan and Bransford 2005). An instructional environment that supports the development of understanding and fluency should be based on a logical progression of content that is connected across grades as well as within grades, should provide opportunities for students and teachers to engage in mathematically substantive discussions, and should involve teachers and students in interpreting and creating mathematical representations to enhance their understanding.

An instructional progression approach

An instructional progression of concepts and skills supports coherence across and within grades. The table at the beginning of each Focal Point outlines the instructional progression and presents the mathematics suggested for grade 3 within the context of the mathematics suggested for the grades immediately before and after. Teacher study groups can work to identify gaps in the knowledge of their students that might be causing them difficulties with the mathematics in grade 3. In addition, the instructional progression offers a view of the future mathematics in which students will be applying the knowledge and skills learned in grade 3.

In-depth instructional conversations

Students have little opportunity to build understanding in a classroom in which the teacher does all the talking and explaining. A valuable instructional approach is one in which teachers create a nurturing, meaning-making community as students use “math talk” to discuss their mathematical thinking and help one another clarify their own mathematical thinking, understand and overcome errors, and describe the methods they use to solve problems (Fuson and Murata 2007). Such discussions identify commonalities and differences as well as advantages and disadvantages across methods. By having students talk about their informal strategies, teachers can help them become aware of, and build on, their implicit informal knowledge (Lampert 1989; Mack 1990). As the teacher and students learn to listen respectfully to the “math talk” of others, they model, structure and clarify, instruct or explain, question, and give feedback to enhance one another’s learning. As students’ understanding and fluency in various topics increase, the amount and type of class discussion related to each topic will change. In-depth discussion of new topics should begin as more sophisticated, mature discussion of previously encountered topics continues.

Using mathematical drawings

The use of mathematical representations, in particular mathematical drawings, during problem-solving discussions and explanations of mathematical thinking helps listeners better understand the speaker. The use of mathematical drawings as a component of homework and classwork by both students and teachers helps them better understand each other’s thinking and thus provides continual assessment to guide instruction as the teacher addresses issues that arise in such drawings and accompanying talk (e.g., errors or interesting mathematical thinking). Mathematical drawings, rather than show situational details of the real objects involved, focus on the mathematically important features and relationships in the situation, such as the quantities and operations, and can involve small circles or other simple shapes to represent a variety of objects. These representations of individual objects can then evolve into schematic numerical drawings that show relationships between, or operations on, numbers rather than show all the individual objects. Examples of mathematical drawings that can be produced and understood by students are used throughout this publication.

An Important Grade 3 Issue: Developing Fluency in Basic Multiplication and Division Facts

The grade 3 Focal Point on multiplication and division suggests that a solid understanding of multiplication gives students the foundation they need to learn, retain, and apply the whole-number multiplication and division facts, and that by the end of grade 4, students should become fluent in using those facts. The related section of this publication presents strategies that help students learn the basic facts in meaningful contexts. Students should be encouraged to discuss those strategies, as well as any other strategies that they construct on their own. The advantage of encouraging students to think about as many diverse strategies as possible is that students make connections among the facts, and those connections lead students to faster, more permanent retention. In general, mathematicians and mathematics educators agree that if students are asked to memorize multiplication and division facts as isolated entities without being supported by meaningful constructs, their ability to develop fluency with those facts and apply them will be limited.

The applications of the basic multiplication and division facts permeate the mathematics that students learn in grade 3 and in later grades. For example, students use those facts in multidigit multiplication and multidigit division, and as building blocks for fraction, decimal, and geometry concepts. Therefore, much attention should be dedicated to teaching, practicing, and discussing a wide variety of strategies for understanding the basic multiplication and division facts, with the eventual goal being the automatic recall of those facts.

The three grade 3 Focal Points and their Connections are reproduced on the following page.

Curriculum Focal Points and Connections for Grade 3

The set of three curriculum focal points and related connections for mathematics in grade 3 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

<table>
<tr><th>Grade 3 Curriculum Focal Points</th><th>Connections to the Focal Points</th></tr>
<tr><td>Number and Operations and Algebra: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts

Students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal “jumps” on number lines for multiplication, and successive subtraction, partitioning, and sharing for division). They use properties of addition and multiplication (e.g., commutativity, associativity, and the distributive property) to multiply whole numbers and apply increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving basic facts. By comparing a variety of solution strategies, students relate multiplication and division as inverse operations.</td><td rowspan="3">Algebra: Understanding properties of multiplication and the relationship between multiplication and division is a part of algebra readiness that develops at grade 3. The creation and analysis of patterns and relationships involving multiplication and division should occur at this grade level. Students build a foundation for later understanding of functional relationships by describing relationships in context with such statements as, “The number of legs is 4 times the number of chairs.”

Measurement: Students in grade 3 strengthen their understanding of fractions as they confront problems in linear measurement that call for more precision than the whole unit allowed them in their work in grade 2. They develop their facility in measuring with fractional parts of linear units. Students develop measurement concepts and skills through experiences in analyzing attributes and properties of two-dimensional objects. They form an understanding of perimeter as a measurable attribute and select appropriate units, strategies, and tools to solve problems involving perimeter.

Data Analysis: Addition, subtraction, multiplication, and division of whole numbers come into play as students construct and analyze frequency tables, bar graphs, picture graphs, and line plots and use them to solve problems.

Number and Operations: Building on their work in grade 2, students extend their understanding of place value to numbers up to 10,000 in various contexts. Students also apply this understanding to the task of representing numbers in different equivalent forms (e.g., expanded notation). They develop their understanding of numbers by building their facility with mental computation (addition and subtraction in special cases, such as 2,500 + 6,000 and 9,000 – 5,000), by using computational estimation, and by performing paper-and-pencil computations.</td></tr>
<tr><td>Number and Operations: Developing an understanding of fractions and fraction equivalence

Students develop an understanding of the meanings and uses of fractions to represent parts of a whole, parts of a set, or points or distances on a number line. They understand that the size of a fractional part is relative to the size of the whole, and they use fractions to represent numbers that are equal to, less than, or greater than 1. They solve problems that involve comparing and ordering fractions by using models, benchmark fractions, or common numerators or denominators. They understand and use models, including the number line, to identify equivalent fractions.</td></tr>
<tr><td>Geometry: Describing and analyzing properties of two-dimensional shapes

Students describe, analyze, compare, and classify two-dimensional shapes by their sides and angles and connect these attributes to definitions of shapes. Students investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students understand attributes and properties of two-dimensional space and the use of those attributes and properties in solving problems, including applications involving congruence and symmetry.</td></tr>
</table>

Reprinted from *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (Reston, Va.: National Council of Teachers of Mathematics, 2006, p. 15).

2 Focusing on Multiplication and Division

In grade 3, students are introduced to the formal operations of multiplication and division. The focus is on understanding the operations and on finding patterns and strategies related to certain factors, with the eventual goal of gaining fluency in using all basic multiplication and division facts. Students gain this understanding and fluency by being able to represent and interpret multiplication and division in a variety of ways, including equal groups, arrays, and area models, and by developing an understanding of the mathematical relationship between multiplication and division.

Instructional Progression for Multiplication and Division

The focus on multiplication and division in grade 3 is supported by a progression of related mathematical ideas before and after grade 3, as shown in table 2.1. To give perspective to the grade 3 work, we first discuss some of the important ideas that students focused on in grade 2 that prepare them for multiplication and division in grade 3. At the end of the detailed discussion of this grade 3 Focal Point, we present examples of how students will use the multiplication and division understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, see the appropriate grade-level books, *Focus in Grade 2* and *Focus in Grade 4*.

Table 2.1 represents an instructional progression for the understanding of multiplication and division in grades 2 through 4.

Early Foundations in Multiplication

In grade 2, students have experienced skip counting, the beginning of the foundation of understanding multiplication as the joining of equal groups. For example, when students in grade 2 skip count groups of two things by 2s, they say 2, 4, 6, 8, 10. At this level, we emphasize that each number in this sequence is 2 more than the number before it. In grade 3, we want students to learn that as they count 2, 4, 6, 8, 10, the numbers they are saying are the totals for different numbers of groups of 2, or different multiples of 2. They are naming the total for one group of 2, two groups of 2, three groups of 2, four groups of 2, five groups of 2, and so on. This idea of "joining equal groups" is the basic meaning for whole-number multiplication.

Students in grade 2 also prepare for multiplication by observing patterns on the hundreds grid. For example, students can highlight the results of joining groups of 10, or skip counting by 10s, as in figure 2.1, to see the pattern of 0s in the ones place.

Table 2.1

Grade 3: Focusing on Multiplication and Division—Instructional Progression for Developing Understanding in Grades 2–4

Grade 2	Grade 3	Grade 4
Students use the base-ten numeration system and place-value concepts to represent numbers up to 1,000. Students use place value and properties of addition and subtraction to compose and decompose multidigit numbers.* Students use addition and subtraction to answer questions about joining and separating situations involving equal groups.* Students skip count by various numbers to build foundations for understanding multiples and factors.*	Students use equal-group, array, and area situations and models to represent and interpret multiplication and division and their relationship. Students use multiplication and division to solve problems. Students connect skip counting with the multiples of a number. Students use patterns in lists of multiples (e.g., in a multiplication table) to learn basic facts. Students use properties of addition and multiplication (e.g., commutative, associative, and distributive properties) and known facts to find related unknown facts. Students use the mathematical relationship between multiplication and division to view "finding a quotient" as "finding an unknown factor."	Students work toward quick recall of basic multiplication facts and related division facts. Students expand their application of multiplication and division to solve problems, including those involving scalar comparison and combination situations. Students use their knowledge of multiples to extend whole-number division to include division with remainders.** Students use place-value patterns to find multiples of tens, hundreds, and thousands. Students apply their understanding of representations for multiplication (i.e., equal groups, arrays, area models, and scalar comparisons), knowledge of basic facts, place value, and the distributive property to multiply multidigit numbers. Students use properties of multiplication and patterns in place value to estimate and calculate products mentally. Students use properties to apply the standard algorithmic approach to multiplication with multidigit numbers and use this approach to solve problems.

* Appears in the Grade 2 Connections to the Focal Points

** Appears in the Grade 4 Connections to the Focal Points

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Fig. 2.1. Pattern for tens on a hundreds grid

Work in grade 2 related to addition "across a decade" is also helpful for developing speed and accuracy in determining an unknown fact from a known fact. For example, in counting by 4s, students can go from 28 to 32 by thinking 28 + 4, 28 + 2 + 2, 30 + 2, 32. Similarly, when counting by 6s and going from 48 to 54, students can think 48 + 6, 48 + 2 + 4, 50 + 4, 54. Ease in making these transitions from twenties to thirties, to forties, to fifties, and so forth, will help students develop fluency in using basic multiplication and division facts.

Building Depth of Understanding in Multiplication and Division

Meaning of multiplication

Multiplication is first introduced as the joining of equal groups. Students learn that the total in 3 groups of 4, or 4 taken 3 times, is written as $3 \times 4 = 12$. The numbers being multiplied are the factors; the answer is the product. Figure 2.2 shows the equal groups model for multiplication. To model $3 \times 4 = 12$, students can draw 3 groups with 4 squares in each group. The product is the total number of objects.

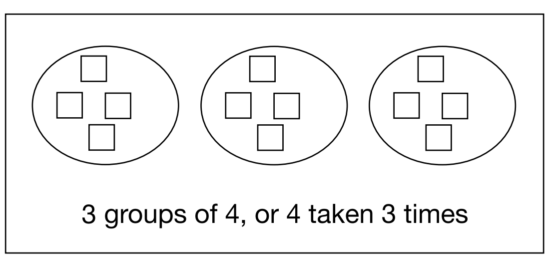


Fig. 2.2. Model of multiplication as joining equal groups

As students gain a deeper understanding of the equal-groups model, they can begin to move away from using objects to show equal groups and counting the objects to the more sophisticated model of writing the number 4 to represent each group and then using skip counting by 4 to determine the product. This move to a more abstract model of equal groups is an important one in students' conceptual development of the meaning of multiplication.

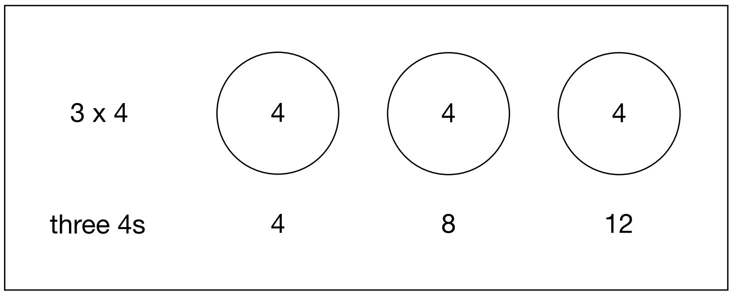


Fig. 2.3. Relating skip counting to a model of equal groups

Note that people in different parts of the world have different conventions about how to read 3 × 5. People in the United States have a nonmandatory interpretation of 3 × 5 as 3 groups of 5, but it could just as easily be read as "3 taken 5 times." Children coming from different places may use opposite conventions, so these different interpretations need to be recognized and discussed early on. To facilitate conversation about a particular equation, a helpful tactic might be to draw a circle around the factor that is being used to describe the size of the group. However, in the long run students benefit from being flexible in their interpretations of the two meanings of the factors.

Another way to model multiplication as the joining of equal groups is with an array. An array is a group of objects arranged in rows of equal length and columns of equal height. These rows and columns show the number of equal groups and the number in each group (the factors). The product is the total number of objects in the array. Students can interpret the array in figure 2.4 by looking at the rows to see 3 rows of 4 (3 groups of 4). By looking at the columns of the same array, students can also interpret the array as 4 columns of 3 (or 4 groups of 3). Students may skip count to find the total number of objects in the array. For example, in figure 2.4, they may skip count by the number in each row, 4, saying 4, 8, 12, and then know that 3 rows of 4 is 12, or 3 × 4 = 12.

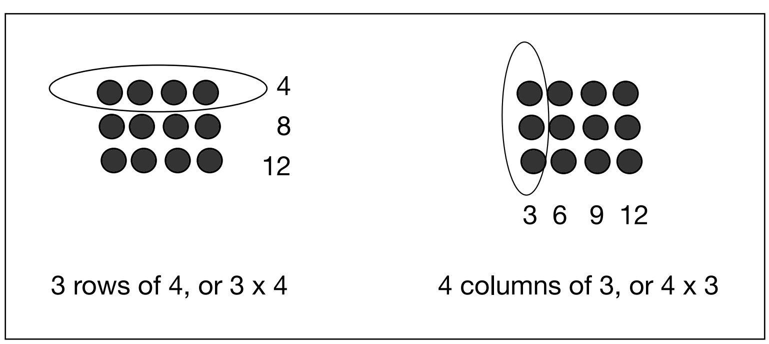


Fig. 2.4. Different interpretations of an array

Even though 3 groups of 4 are physically different from 4 groups of 3, the array model shows that mathematically, in multiplication, it does not matter which factor represents the groups and which factor represents the quantity in each group. This property of multiplication is called the *commutative property;* multiplying factors in any order results in the same product.

In the area model, the squares in an array are pushed together into a rectangle, as in figure 2.5. Students can also relate the area model to the equal-groups model by circling groups of rows or groups of columns.

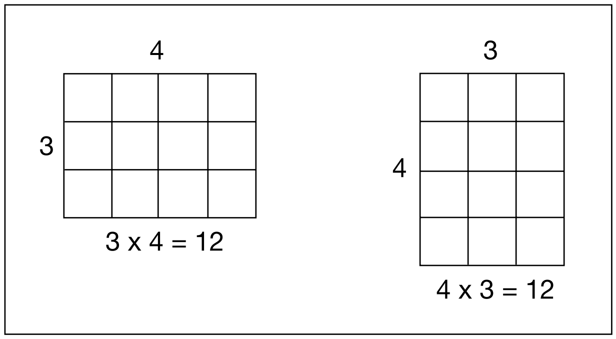


Fig. 2.5. Model of multiplication as area of a rectangle

Using area models is an effective way to demonstrate in general the commutative property of multiplication because the area model connects with both arrays and equal groups. By rotating the 3 × 4 rectangle having 3 rows of 4 squares, students can see that it is the same as a 4 × 3 rectangle having 4 rows of 3 squares. The rectangle itself did not change, so the total area is still 12 unit squares, which can be represented by 3 × 4 or 4 × 3.

Using a variety of models to represent multiplication and exploring connections among the models will enhance students' understanding of the meaning of multiplication. This understanding will facilitate their learning of basic multiplication facts.

Strategies for learning multiplication and division facts

A solid understanding of the meaning of multiplication will give students the foundation they need to learn, retain, and apply the basic multiplication facts. Many strategies involving the meaningful application of the properties of multiplication can be used to help students find products. The area models from figure 2.5 provide a pictorial example of the commutative property of multiplication. They show that 3 × 4 = 4 × 3 = 12. The commutative property of multiplication states that two factors can be multiplied in any order to get the same product. Applying this understanding of the commutative property to all numbers will help students learn their multiplication facts at a much faster pace because the number of facts that students need to learn is decreased almost by half (see table 2.2).

The associative property of multiplication also can be used by students to learn new facts. The associative property of multiplication, when combined with the commutative property, allows more than two factors to be multiplied in any combination of two factors in any order. In particular, the associative property can be used to apply the idea of doubling to learn new facts. For example, if students know that 2 × 7 = 14, they can use doubling to find 4 × 7. Students who know that 4 is double 2 can think of the product 4 × 7 as (2 × 2) × 7, or 2 × (2 × 7). Since 2 × 7 = 14, then 4 × 7 is two of that product, and 14 + 14 = 28. Figure 2.6 gives an example of doubling.

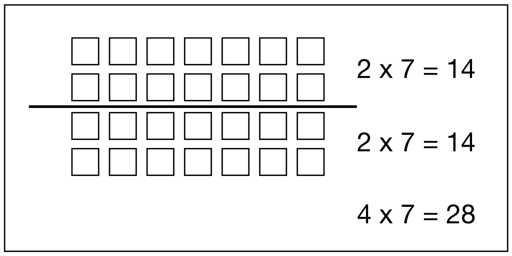


Fig. 2.6. Using the associative property of multiplication to double a known fact

In grade 3, students are introduced to the distributive property of multiplication over addition as a strategy for using products they know to find products they do not know. The distributive property of multiplication over addition is expressed mathematically as $a \times (b + c) = (a \times b) + (a \times c)$. In other words, a factor can be "taken apart" into addends and the addends can each be multiplied by the other factor to find partial products, and then those partial products can be added to find the total product. For example, in figure 2.7, to find 8 × 6, students who know the multiplication facts for 5 can think of 6 as 5 + 1. Then they can multiply 8 × (5 + 1). Through the distributive property, they can "distribute" the 8 over the 5 + 1 and rewrite the expression 8 × 6 as (8 × 5) + (8 × 1). Then they can use the facts they know, that 8 × 5 = 40 and 8 × 1 = 8, and can then add 40 + 8 to get 48. Thus, they have used the distributive property of multiplication over addition to determine that 8 × 6 = 48.

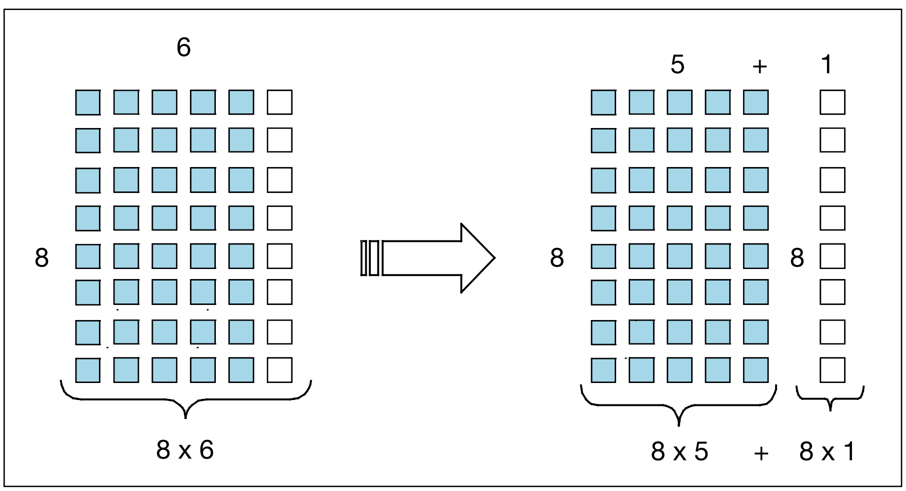


Fig. 2.7. Picturing the distributive property of multiplication over addition:
8 × 6 = 8 × (5 + 1) = (8 × 5) + (8 × 1)

Area models can also be used to illustrate the use of the distributive property of multiplication over addition to build unknown products from known products. For example, figure 2.8 shows a model of 7 × 8. The area model is divided into two sections: one section that represents 5 × 8 and another section that represents 2 × 8, because 7 can be thought of as the sum 5 + 2. Students can join the areas from both sections and add the areas to get a total product of 56. Thus, they can see from the model that 7 × 8 = (5 × 8) + (2 × 8) = 40 + 16 = 56. This strategy is more efficient than counting the squares in a 7 × 8 model to find the product of 56.

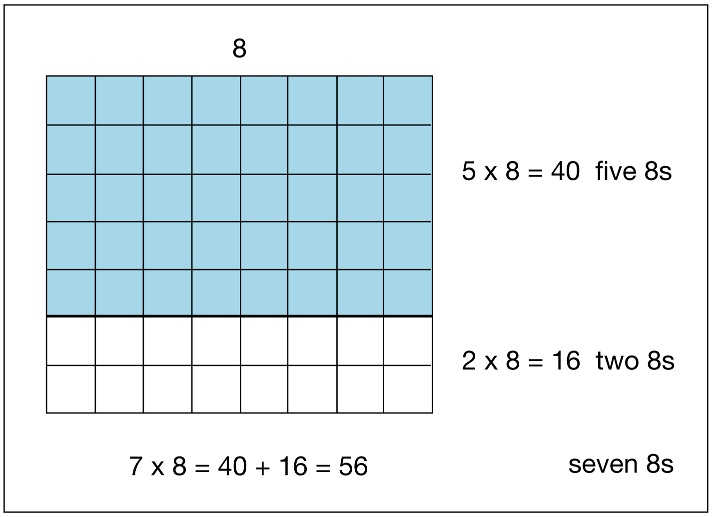


Fig. 2.8. An area model showing the distributive property of multiplication over addition

Patterns in a multiplication table can help students develop the conceptual groundwork needed for learning basic multiplication facts. For example, in figure 2.9, the multiples of 9 are highlighted. Students may notice how the multiples of 9 can be found across a row and down a column on a multiplication table. At first, students may identify patterns in the row and column—for example, the tens digit increases by one when reading across or down, and the ones digit decreases by one when reading across or down. Students may notice that the difference between adjacent values in the 10 column and the 9 column is the number that is multiplied by 10 to produce the value in the 10 column. For example, 70 = (7 × 10); 70 – 7 = 63 = 7 × 9.

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 2.9. Multiplication table

Table 2.2 summarizes some of the strategies that can be used when multiplying with certain factors.

Table 2.2
*Some Strategies for Multiplying**

Multiplying by 0	The product is always 0. For example, $0 \times 2 = 0$.
Multiplying by 1	The product is always the other factor. For example, $1 \times 5 = 5$.
Multiplying by 2	You can skip count by 2s.
Multiplying by 3	You can multiply by 2, creating two groups, and add one more group. For example 3×7 is 2×7 plus one more 7.
Multiplying by 4	You can multiply by 2 and then double the product. For example, $4 \times 7 = 2 \times 2 \times 7 = 2 \times 14$. Two 14s are 28.
Multiplying by 5	You can skip count by 5s. Or you can multiply by 10 and take half of the product. For example 8×5 is half of 8×10, or half of 80, which is 40.
At this point, students can use the commutative property to realize that they already know many of the facts for greater factors. For example, when multiplying by 6, 6×4 is the same as 4×6.	
Multiplying by 6	You can multiply by 5, creating five groups, and then add one more group. For example, $6 \times 8 = 5 \times 8 + 1 \times 8 = 40 + 8 = 48$.
Multiplying by 7	You can multiply by 5, creating five groups, and then add two more groups. For example, $7 \times 8 = 5 \times 8 + 2 \times 8 = 40 + 16 = 56$.
Multiplying by 8	The only 8s that you cannot apply a previous strategy to are 8×8, 8×9, and 8×10. The product of 8×8 is 4×8 doubled, or two 32s, which is 64. See below for 8×9 and 8×10.
Multiplying by 9	You can multiply by 10, creating ten groups, and then subtract one group. For example, nine 7s is ten 7s minus one 7, or $9 \times 7 = (10 \times 7) - (1 \times 7)$.
Multiplying by 10	You can count by 10s.

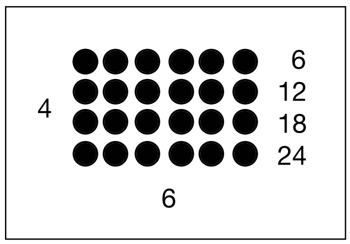
*These strategies and other ideas for learning multiplication facts are discussed in more detail on pages 108–13 of *Navigating through Number and Operations in Grades 3–5* (Duncan et al. 2007).

Strengthening understanding of multiplication through problem solving

Using multiplication to solve problems can help students better understand the operation and its applications. The strategies students use to solve the problems will vary depending on their comprehension of multiplication as well as the extent to which they can recall the basic multiplication facts. For example, a group of students in grade 3 was presented this scenario: "Maria needs to buy invitations for her birthday party. The invitations come in packages of 6. If Maria buys 4 packages, how many invitations will she have?" The following discussion between the teacher and the students reveals that students with varying degrees of understanding solved the problem in different ways and that the discussion of the various strategies contributes to each student's understanding.

Teacher: Angie, how did you solve the problem?

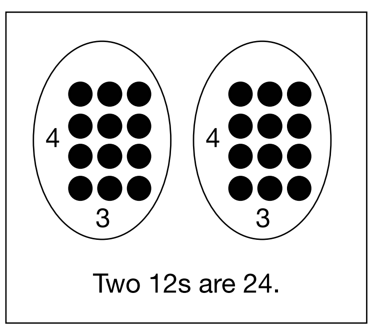
Angie: I drew an array with 4 rows of 6 dots. I then counted rows of dots to find a total of 24. So Maria will have 24 invitations.



Teacher: Angie, how did you count the rows of dots?

Angie: There were 4 rows with 6 in each row, so I skip counted by 6 four times: 6, 12, 18, 24. It was sort of like adding 6 + 6 + 6 + 6. Oh, that's interesting. I didn't really need to draw all of those circles. I could have drawn a loop for each row and just put the number 6 in each one. That's what I'll do the next time to be faster.

Josh: I solved it a different way. I made the same array as Angie, but then I saw that the big array was really two smaller arrays with 3 across. I multiplied 4 × 3 and got 12. There are 12 in each array, and two 12s are 24. Maria has 24 invitations.



Teacher: Josh, why did you split up your array into two smaller arrays?

Josh: I don't know what 4 × 6 is, but I know that 4 × 3 = 12, so I knew I could find that product and double it to get the answer.

Aissa: I did not draw a model because I knew about the property that says you can take one of the factors apart: $4 \times 6 = 4 \times (3 + 3) = (4 \times 3) + (4 \times 3)$; $4 \times 3 = 12$ and $12 + 12 = 24$. Maria has 24 invitations.

Teacher: Aissa, why did you change the factor of 6 into 3 + 3?

Aissa: I did not know 4 × 6, but I did know 4 × 3, so I changed the 6 into two 3s and multiplied and then put it back together.

Carly: I did it a different way. I know that 4 packages of 6 invitations means 4 groups of 6, which is multiplication. So I multiplied 4 × 6 and got 24. Maria has 24 invitations.

Teacher: Carly, why didn't you use a model or a property?

Carly: I knew I needed to multiply 4 × 6, and I knew that 4 × 6 = 24.

Teacher: It is interesting that Angie, Josh, Aissa, and Carly all got the same answer, but each got their answer a different way. Is one way better than another way?

Theo: Well, you can solve the problem using any of their ways, but Carly's way is the fastest. It is faster to know the answer than to draw an array, skip count, or even use a property.

Teacher: That is true, but all the strategies arrive at the same answer because they are all correct, and they are all related. Angie's strategy used an array to show the meaning of multiplication as joining equal groups and connected this meaning to skip counting. And then she noticed that she could represent an array with just numbers and did not have to draw all the circles. Josh and Angie both used arrays, but Josh recognized in the array a multiplication fact that he knew: 4 × 3 = 12. By using the array and this multiplication fact, he was able to determine that 4 × 3 = 12 and two 12s are 24. Aissa used the same numbers that were in Josh's strategy and applied the distributive property of multiplication over addition. She realized without drawing pictures that 6 = 3 + 3, and so she could multiply 4 times each addend in 3 + 3 to find that 4 × (3 + 3) = (4 × 3) + (4 × 3) = 12 + 12 = 24; so 4 × 6 = 24. Carly understood from the situation in the problem that she was joining equal groups and needed to multiply 4 times 6. She knew that 4 × 6 = 24, so she used that product to solve the problem.

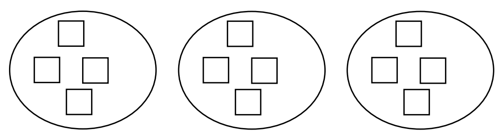
As the foregoing discussion shows, less confident students may rely more at first on visual representations to solve multiplication problems. Students who have developed a greater understanding of multiplication may use known facts coupled with the implementation of a strategy, such as doubling, to find solutions. An important aspect to note is that during the discussion, the teacher prompted students to explain their thinking and guided them through the progression of the different strategies that each student used. Students benefit from connecting the early representations using models with the more sophisticated strategies.

Understanding relationships among the models and strategies used to solve problems involving multiplication leads students to a deeper understanding of the operation and its properties. If students implement their strategies effectively, they will be more likely to find correct solutions. However, one of the ultimate goals of this Focal Point is for students to develop fluency in their knowledge of basic multiplication facts so that they can solve problems more efficiently. Although students need to move from concrete models to more advanced strategies to automatic recall of the multiplication facts, the pace at which students move will vary, depending on their understanding of the meaning of multiplication and their facility for remembering.

Using models to connect multiplication and division

The use of visual representations is as vital to students' understanding of division as it is to their understanding of multiplication. Multiplication problems arise from joining equal groups of objects; division problems arise from separating a set of objects into equal groups. The use of familiar models along with students' understanding of multiplication forms the framework for developing understanding of division. Models using equal groups, arrays, and area can all help students in their development of the concept of division and the mathematical relationship between multiplication and division. Students do not need to be fluent in using multiplication facts before they can explore this relationship. Multiplication and division are closely related mathematically and should be studied simultaneously. Division facts then become not something new to learn, but simply a reinterpretation of the multiplication facts.

Students develop an understanding of division as the inverse of multiplication by separating a whole into equal groups. Because the roles of the factors in equal-group situations are different, two division situations are possible, depending on whether the number of groups or the size of the groups is known. Students have seen the situation in figure 2.10 as 3 equal groups with 4 in each group and have used it to represent the question "If there are 3 groups of 4, how many are there in all?" The answer to this question is represented by $3 \times 4 = 12$. To learn about the two types of division situations, students can use the arrangement of 12 separated into 3 equal groups with 4 in each group to answer the question "If 12 is divided into 3 equal groups, how many are in each group?" The division equation that represents the answer to this question is $12 \div 3 = 4$. Alternatively, students can use the representation of 12 separated into 3 equal groups with 4 in each group to answer the question "If 12 is divided into equal groups of 4, how many groups are there?" The answer to this question is represented by $12 \div 4 = 3$.



Separating 12 into 3 equal groups gives 4 in each group: 12 ÷ 3 = 4 because 3 x 4 = 12.
Separating 12 into equal groups of 4 gives 3 groups: 12 ÷ 4 = 3 because 3 x 4 = 12.

Fig. 2.10. Division as separating into equal groups

Arrays and area models also help students connect the concepts of multiplication and division. As a result of their work in multiplication, students can observe that the product of the number of rows (width) and the number of columns (length) in an array or area model gives the total number of squares in the array or rectangle. As shown in figure 2.11, students can apply this understanding as they use this model to find $36 \div 4$. Interpreted as an array, the model shows a total of 36 squares in 4 equal rows. The quotient (unknown factor) is the number of squares in each row, which is the number of columns. Interpreted as an area model, the model shows a rectangle with an area of 36 square units and a width of 4 units. The quotient (unknown factor) is the number of units in the length. In either situation, the model shows $36 \div 4 = \square$, as well as the related unknown factor problem, $4 \times \square = 36$. Students can use the model or count up by 4s to 36; or see by the pattern in its digits that 36 is a multiple of 9; or think $4 \times 9 = 36$ to determine that the quotient, or unknown factor, is 9.

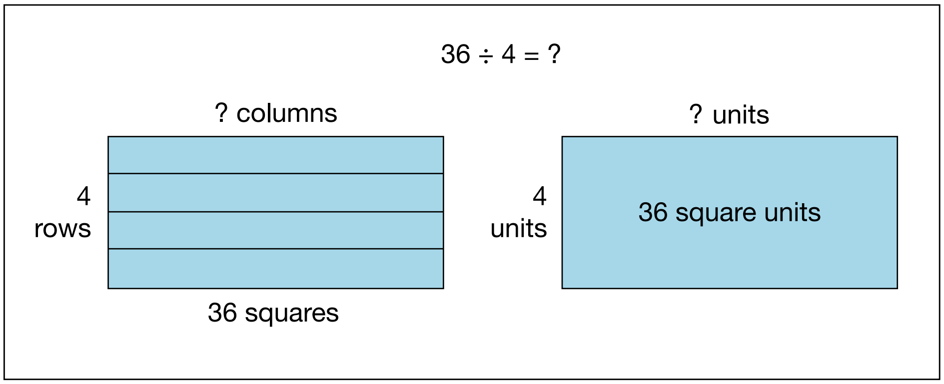


Fig. 2.11. Division as finding an unknown number of columns in an array or an unknown side length in a rectangular area

Connecting models for multiplication with models for division supports students' understanding of the mathematical relationship between the two operations. It also helps lead students to the realization that the division facts are really multiplication facts with "unknown" factors, and that, if they know the multiplication facts, they actually already know the division facts.

Using language to connect multiplication and division

The use of parallel language in multiplication and division also facilitates students' understanding of the mathematical relationship between the two operations. If students are asked to identify the meaning of $6 \times 3 = 18$, they should understand that "6 groups of 3 are 18 in all." As students develop an understanding of division, they need to identify the meaning of $18 \div 6 = 3$ as 18 divided into 6 groups to give 3 in each group or as 18 divided into groups of 6 to give 3 groups in all. When students learn to interpret division in these ways, they realize that knowing $6 \times 3 = 18$ can help them determine the quotients $18 \div 6 = \square$ and $18 \div 3 = \square$. Because of the connection between multiplication and division, students can identify and learn the basic division facts from the known multiplication facts. For example, if students know that $8 \times 5 = 40$ and $5 \times 8 = 40$, and they understand the relationship between multiplication and division, then they actually already know that $40 \div 5 = 8$ and $40 \div 8 = 5$. As students' understanding of related multiplication and division facts matures, they begin to realize the relationship between the number sentences as shown in figure 2.12—that the product in a multiplication fact becomes the dividend in the related division fact, and the factors become the divisor and quotient.

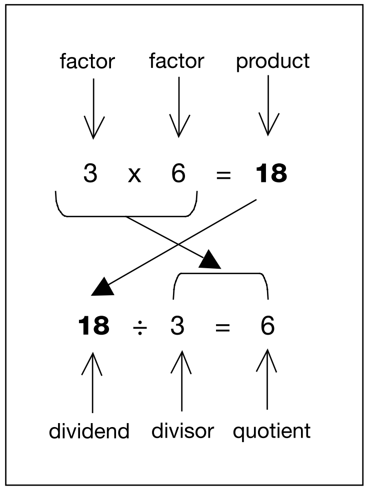


Fig. 2.12. The relationship between the terms in multiplication and division equations

Using strategies to connect multiplication and division

Students in grade 3 can also apply their knowledge of multiples and skip counting when learning division. As students skip count, they often keep track of the multiples on their fingers. For example, to find 6 × 5, students skip count by 5s six times, putting up a finger for each 5. When the students have put up six fingers, they have reached 30, so they know that 6 × 5 = 30. In this example, the number that the students say when they raise the sixth finger is the answer. This same counting strategy can be applied to division. Students can find 30 ÷ 5 by thinking, "How many fives are in 30?" (? × 5 = 30) and then counting by 5s up to 30, putting up one finger for each 5. In this example, the answer is the number of fingers the students have up when they say 30. Thus, students can observe that they have six fingers up, so there are six 5s in 30, and 30 ÷ 5 = 6. Students have answered the question "What is 30 divided by 5?" by answering the question "What number times 5 equals 30?" They have used their understanding of multiplication to divide. This method of skip counting and its relationship to multiplication helps students see finding the quotient as "finding an unknown factor," which further emphasizes the relationship between multiplication and division.

Connecting multiplication and division through models, language, and strategies will help lay the groundwork for understanding division and its mathematical relationship to multiplication. For further examples and ideas on the mathematical relationship between multiplication and division, refer to pages 117–20 in *Navigating through Number and Operations in Grades 3–5* (Duncan et al. 2007).

Strengthening understanding of division through problem solving

As mentioned previously, students solve two types of equal-groups division problems in grade 3:

- "How many in each group?" or partitive division problems
- "How many groups?" or measurement division problems

Although students do not learn the names of these types of division situations, they should be able to analyze and describe the similarities and differences between them. Allowing students opportunities to model and solve the two different kinds of division problems in a variety of different contexts will deepen students' understanding of division. Consider the examples that follow.

"How many in each group?": partitive division problems

Sarah is making 3 bracelets. Each bracelet will have the same number of beads. If Sarah has 24 beads, how many beads will she use for each bracelet? To solve this problem, students can "deal out" the 24 beads to partition them into 3 groups. They start by putting 1 object at a time in each group. They continue the partitioning, "dealing out" the objects into equal groups until none are left. The quotient is the number of objects in each group. As can be seen from figure 2.13, 24 ÷ 3 = 8, so Sarah will put 8 beads on each bracelet.

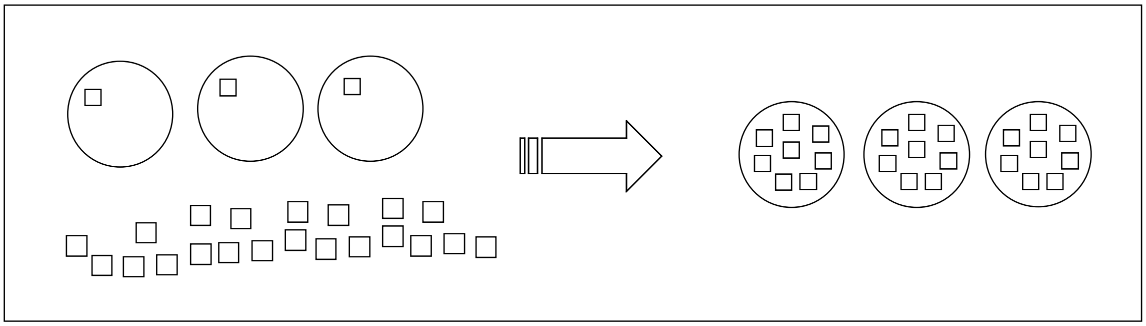


Fig. 2.13. Partitive model to show that 24 beads divided into 3 groups give 8 beads in each group: 24 ÷ 3 = 8

"How many groups?": measurement division problems

Contrast the previous situation with this one: Sarah is making bracelets. She wants to put 3 red beads on each bracelet. If Sarah has 24 red beads, how many bracelets can she make with the red beads? To solve this problem, students can use a grouping, or measurement, model. They start with 24 objects. They make 1 group of 3, then another group of 3, then another, until they have used all 24 objects. This act of using a group of 3 over and over again to "measure" the larger group of 24 is similar to laying down a unit of length over and over again and counting how many units have been used to measure a length. The quotient is the number of groups of 3 that students can make from 24. As can be seen from figure 2.14, 24 ÷ 3 = 8, and Sarah will be able to make 8 bracelets with 3 red beads on each.

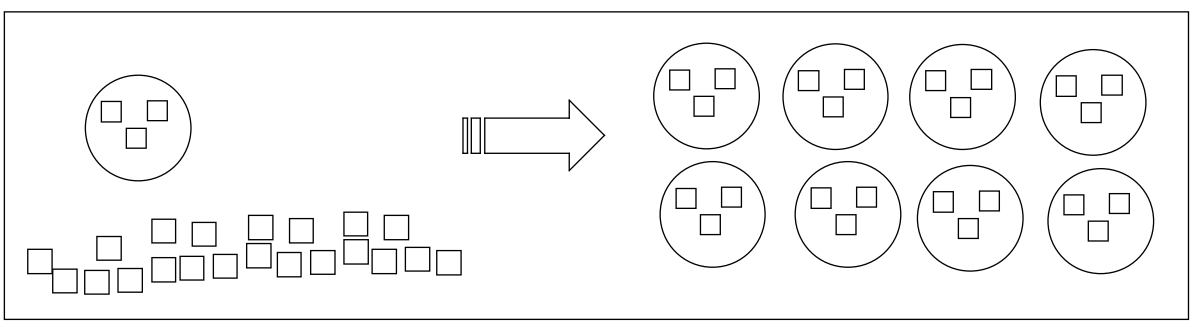


Fig. 2.14. Measurement model to show that 24 beads divided into groups of 3 make 8 groups: 24 ÷ 3 = 8

As is apparent in figures 2.13 and 2.14, the situations are different, but the division equations used to represent the problems are the same. In the partitive model, students deal out objects into 3 equal groups to find the number in each group. In the measurement model, students separate out equal groups of 3 objects to discover how many equal groups are possible. Figure 2.15 demonstrates the connections between these two types of division problems and their subsequent models. In the partitive model, every time students "deal out" 1 object to each row, they deal out a group of 3 objects. They end up dealing out a group of 3 objects 8 times. Similarly, in the measurement model, as students separate the objects into columns of 3, they are also making groups of 3, and they end up making a group of 3 objects 8 times. In both of these situations, the "groups of 3 made 8 times" is reflected in the related multiplication equation, 8 × 3 = 24.

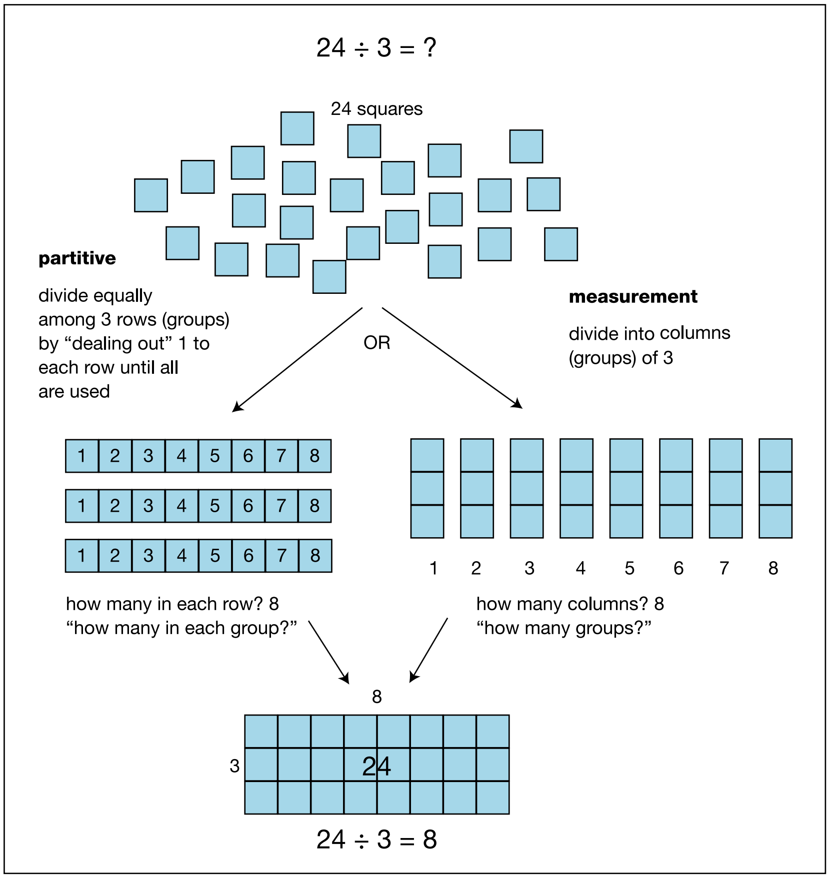


Fig. 2.15. Dividing into 3 groups or groups of 3 to see that 24 ÷ 3 = 8

Strengthening Fluency in Multiplication and Division through Connections

Students must be exposed to and understand connections between mathematical concepts. These connections lead to a deeper understanding of mathematics and how concepts build on one another and are related both within a grade level and across several grade levels. Connections with other mathematical concepts at this grade level are evident throughout this Focal Point.

Previous examples addressing the commutative, associative, and distributive properties illustrate opportunities to build connections between multiplication and division and algebraic reasoning. In addition, an understanding of rule-driven patterns is integral to developing algebra readiness. Describing, predicting, and extending patterns when skip counting or analyzing numbers on a multiplication table help develop an understanding of certain types of functional relationships, an example of which is shown in the chart in figure 2.16, which presents the relationship between the number of automobiles and the total number of wheels on the automobiles. This learning will build the foundation for later understanding of linear relationships. Once students have developed an understanding of functional relationships of this type, they can begin to describe them in context with such statements as, "One year of a dog's life is like 7 years of a person's life."

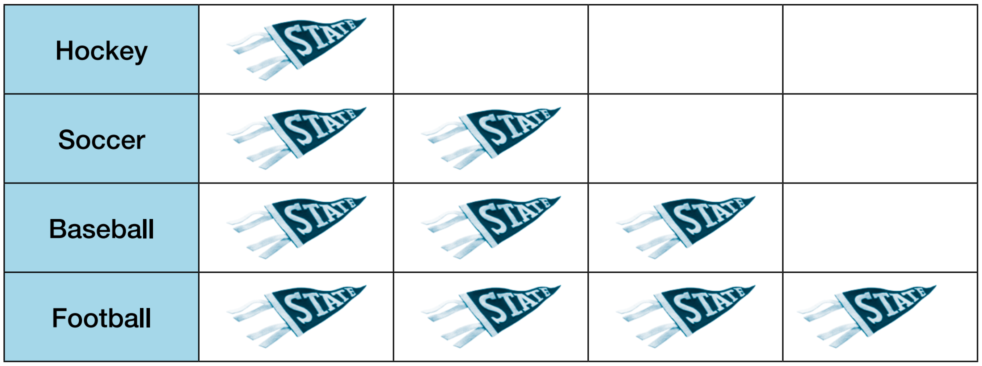
Number of Automobiles	Total Number of Wheels
0	0
1	4
2	8
3	12
n	$n \times 4$

Fig. 2.15. Chart showing the functional relationship between the number of automobiles and the total number of wheels

The connection between multiplication and measurement is also evident in this Focal Point. Area models for multiplication (see fig. 2.5) illustrate a direct connection between multiplication and measurement by exposing students to a model of a rectangle made from square units. As students' understanding of area models progresses, they can find a product by knowing that 5 rows of 3 square units is the same as 5 groups of 3, or $5 \times 3 = 15$. When students reach grade 4, understanding area models will enable them to use the areas of rectangles to model multidigit multiplication. Students can connect the lengths of the sides of a rectangle with the factors being multiplied to confirm the area of the rectangle.

Connecting multiplication with data analysis is another meaningful way to extend students' understanding of multiplication. Interpreting frequency tables, line plots, bar graphs, and picture graphs to solve problems involving multiplication allows students to apply their understanding of multiplication to other strands of mathematics. For example, students can use multiplication to interpret the pictograph shown in figure 2.16, in which a pennant icon stands for 7 votes for a favorite sport. Students could use their understanding of division to create pictographs of this type.

Favorite Sports of a Third-Grade Class



Key: 1 pennant = 7 votes

Fig. 2.16. Using multiplication and division to represent and analyze data

Connections in later grades

As students in grade 3 learn about the meanings of multiplication and division and build their fluency with basic multiplication and division facts, they are laying the foundation that will enable them to gain the computational skills necessary to solve problems with larger numbers in grade 4. For example, in grade 3, students might use the distributive property of multiplication over addition to decompose 8 into 5 + 3 to multiply 8 × 6:

$$(5 + 3) \times 6 = (5 \times 6) + (3 \times 6) = 30 + 18 = 48$$

As students use larger numbers and progress to more complicated problems in grade 4, they can learn to apply the same strategy. For example, to multiply 78 × 6, they might decompose 78 this way:

$$78 \times 6 = (70 + 8) \times 6 = (70 \times 6) + (8 \times 6)$$

The underlying concept of using the distributive property to decompose larger numbers as shown in this example is the foundational principle behind the standard algorithmic approach to multiplication in the grade 4 curriculum focal points. (See *Focus in Grade 4* for more discussion of this strategy.)

In grade 4, students also extend their understanding of multiplication to include the idea of scalar comparisons by dealing with situations involving such language as "5 times as many." This comparative interpretation of multiplication allows students to generalize the ideas of multiplication with whole numbers to multiplication with fractions and decimals.

Later, in grade 5, students will extend their learning to solve problems involving multidigit division. They will continue to use those skills throughout later grades and in a variety of mathematical contexts.

Developing Depth of Understanding

After reading about focusing on multiplication and division facts as suggested in the grade 3 Number and Operations and Algebra Focal Point, consider what activities you already do or plan to do with your class to help students understand the meanings of, and connections between, multiplication and division. How will these activities help students deepen their understanding and build fluency?

3 Focusing on Fractions

In grade 3, students are introduced to fractions. The focus for students is on understanding that fractions use the relationship between two numbers to name part of a whole, part of a set or collection of objects, or points and distances on a number line. Students can gain this understanding by using a variety of models to represent fractions, including regions of wholes or area models, set models, and linear models such as number lines. As students develop the meaning of fractions, they need to develop an understanding of what the numbers in the numerator and denominator of a fraction represent. If students have an understanding of the components of a fraction, they should be able to visualize, represent, and communicate with and about fractions.

Instructional Progression for Fractions

The focus on fractions in grade 3 is supported by a progression of related mathematical ideas before and after grade 3, as shown in table 3.1. To give perspective to the grade 3 work, we first discuss some of the important ideas that students focused on in grade 2 that prepare them for work with fractions in grade 3. At the end of the detailed discussion of this grade 3 Focal Point, we present examples of how students will use the fraction understandings and skills that they learn in grade 3 in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, see the appropriate grade-level books, for example, *Focus in Grade 2* and *Focus in Grade 4*.

Table 3.1 represents an instructional progression for developing the understanding of fractions in grades 2–4.

Early Foundations in Fractions

Students begin to build a foundation for understanding fractions as early as grade 1. In grade 1 students begin to understand the relative magnitudes of numbers and the idea that the counting numbers appear in a sequential order. This idea of order is particularly important as students progress to fraction concepts. If students have a sense of order as it relates to counting numbers, then they can extend this understanding to fractions. One method often used in grade 1 to develop number sense is to show numbers in sequential order on the number line. Looking at figure 3.1, students can use the number line to determine that numbers closer to 0 and to the left on the number line are less than numbers closer to 10 and to the right on the number line. They can also use the number line to identify whole numbers that are between other whole numbers.

Table 3.1

Grade 3: Focusing on Fractions—Instructional Progression for Developing Understanding in Grades 1–4

Grades 1 and 2	Grade 3	Grade 4
Students use the number line to represent the sequential order of the counting numbers and their relative magnitudes. (Grade 1) Students use the concept of partitioning (the mental activity of slicing the length of an object into equal-sized units) to demonstrate the meaning and processes of measurement. Students experience the inverse relationship between the size of a unit of measure and the number of units used in a particular measure. Students compose and decompose two-dimensional shapes to develop foundations for understanding fractions.*	Students use unit fractions (1/*n*) to represent equal divisions of a whole. Students create nonunit fractions by joining unit fractions (e.g. 2/3 is the same as 1/3 + 1/3) and build a whole (1) by joining "*n*" of the unit fraction 1/*n*. Students judge the size of a fractional part by relating it to the size of the whole. Students compare unit fractions of the same-sized whole by observing that the greater the denominator, the smaller the amount represented by the unit fraction. Students use fractions to represent numbers that are equal to, less than, or greater than 1. Students compare and order fractions by using models, benchmark fractions, common numerators, and common denominators. Students use models, including the number line, to identify equivalent fractions. Students use fractional parts of units to measure length. **	Students analyze techniques that involve multiplication and division to generate equivalent fractions.*** Students identify equivalent symbolic representations of improper fractions and mixed numbers. Students use decimal notation as an extension of the base-ten system to the right of the ones place. Students use their understanding of fractions and place value to read and write decimals that are greater than 1 or between 0 and 1. Students connect equivalent fractions and decimals by comparing models with symbols and using equivalent symbols to describe the same point on a number line. Students use place-value notation and equivalent fractions to identify equivalent decimals. Students use place-value notation and understanding of fractions to compare and order decimals.

* Appears in Grade 2 Connections to the Focal Points

** Appears in Grade 3 Connections to the Focal Points

*** Appears in Grade 4 Connections to the Focal Points

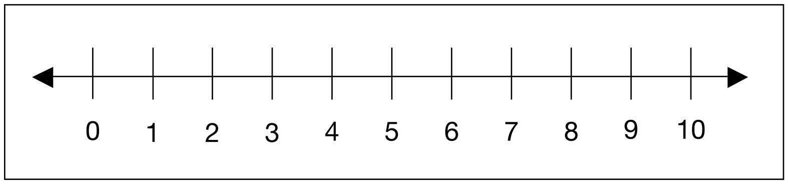


Fig. 3.1. Whole numbers on a number line

As students' understanding of the magnitudes of numbers matures, the number line can also be used to compare numbers. To compare numbers, students may use benchmark numbers, such as 0, 5, and 10. Students should have many concrete and context-related exposures to the benchmark numbers 5 and 10 in both kindergarten and grade 1. For example, figure 3.2 shows the number line with points labeled at 4 and 9. To compare 4 and 9, students may notice that 4 is to the left of the benchmark number 5, so 4 is less than 5. Then they may notice that 9 is to the right of 5, so 9 is greater than 5. Students can then use their reasoning and number sense to determine that if 4 is less than 5 and 9 is greater than 5, then 4 must be less than 9, or $4 < 9$. Conversely, 9 is greater than 4, or $9 > 4$. Students can apply this same reasoning and sense of number when they use number lines to compare fractions.

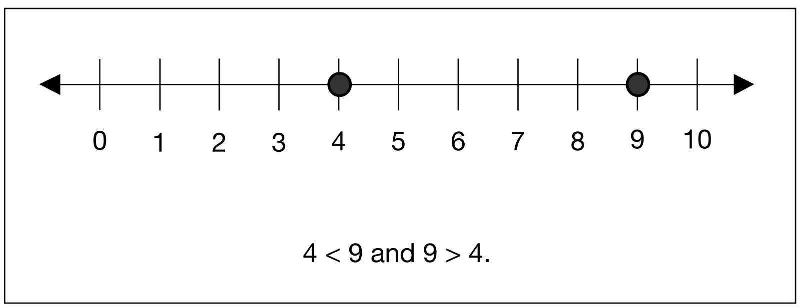


Fig. 3.2. Comparing whole numbers on a number line

Another concept that is vital to students' understanding of fractions is the concept of equality. In early grades, students are shown two sets of objects and must determine whether they show equal quantities. Once students understand symbols, they can connect the sets of objects with counting numbers. Students can then determine equality by using both objects and symbols. See figure 3.3 as an example of equality with and without symbols. Students can use this important concept in the study of fractions. For example, students apply the concept of equality when they determine whether an area model is divided into equal parts.

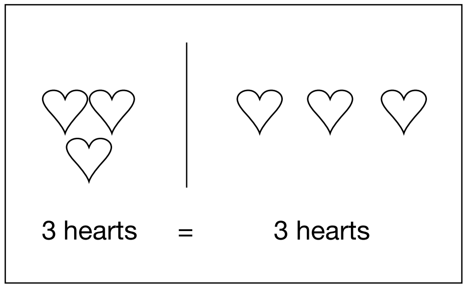


Fig. 3.3. An example of equality with and without symbols

When students reach grade 2, they begin to learn the meaning and processes related to measurement. Underlying these meanings and processes are concepts that will later aid students in their understanding of fractions. The concept of partitioning that students learn in measurement is one such example. Partitioning, which is also a process that occurs in learning about division in grade 3, includes the mental activity of separating an object into equal-sized units. This concept connects with an understanding of equality.

In grade 2, during experiences with linear measure, students can observe that an object's length can be measured if the object is divided into equal-sized units, or if equal-sized units are arranged in a line along the length of the object. These units can be nonstandard units—for example, paper clips, counters, or toothpicks—or standard units—for example, inches or centimeters. The number of these partitioned units yields the measurement of the length of the object. For example, if the length of a pencil can be partitioned equally into seven inch-sized pieces, the length of the pencil is 7 inches. If students have an understanding of equality, they can identify when the partitions of an object are equal. For example, figure 3.4 shows two rectangles that are both divided into five sections. The rectangle on the left represents appropriate partitioning for measuring the rectangle's length because the sections are of equal length. The rectangle on the right does not represent appropriate partitioning for measurement, because the sections are of different length.

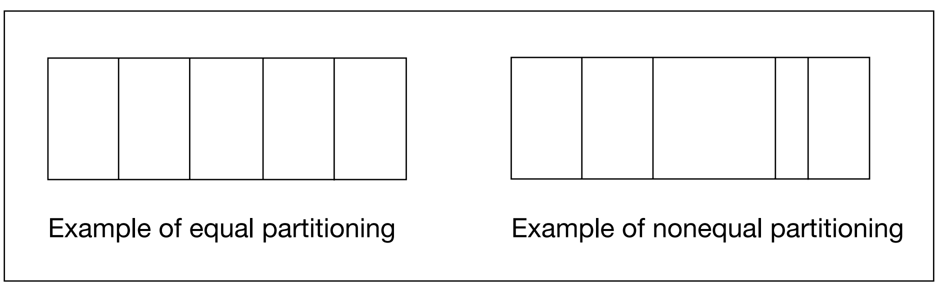


Fig. 3.4. Equal and nonequal partitioning

Students' understanding in grade 2 of the difference between the two examples in figure 3.4 is essential to their development of the foundation of fractions in grade 3, which involves the idea that fraction names are determined by equally sized portions of a whole. This idea of partitioning is also applied in grade 3 work in division, when students are asked the question "If you have 14 objects and you want to make 7 equal-sized groups, how many are in each group?" Students can use the same diagram as in the first part of figure 3.4, dealing out one object per group until all objects are used, resulting in two objects per group (see the multiplication and division Focal Point).

Another important underlying concept in students' grade 2 study of measurement is the inverse relationship between the number of units needed to measure an object and the unit used to measure the object. By building a repertoire of experiences with measuring, students can observe that the larger the unit of measure used to measure an attribute of an object, the smaller the number of units needed in the measurement; inversely, the smaller the unit of measure used, the larger the number of units needed in the measurement. For example, figure 3.5 shows a comparison of the same segment measured in feet and measured in inches. The picture shows that the line is 2 feet long. However, when measured in inches, it is 24 inches long. Because feet are larger units than inches, fewer feet than inches are needed to describe the same length. Conversely, because inches are smaller units than feet, more inches than feet are needed to describe the same length.

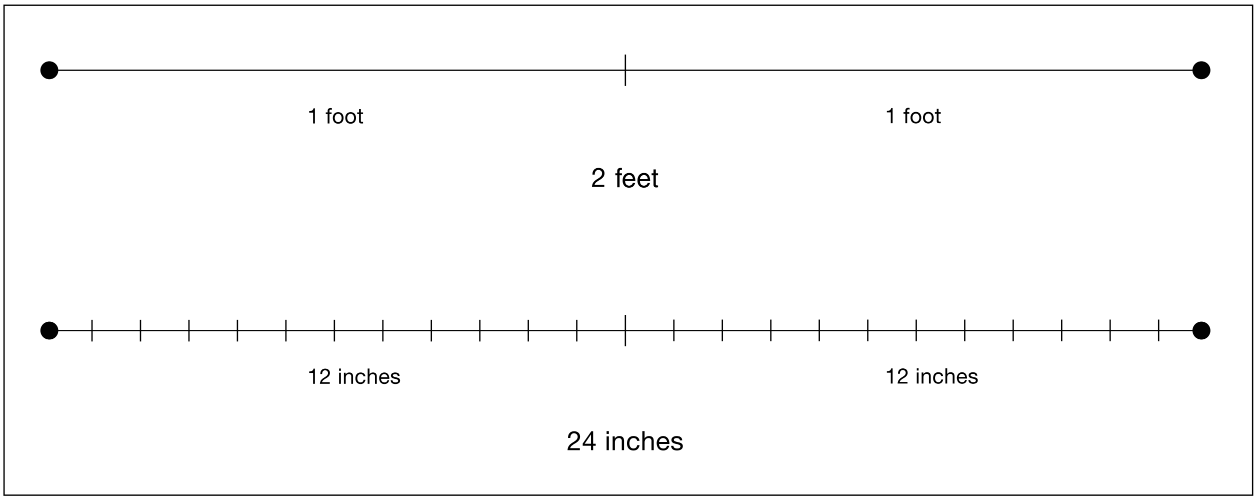


Fig. 3.5. The same length measured in both inches and feet (not drawn to scale)

Having a firm understanding of this inverse relationship between the size of the unit and the number of units needed to measure an object forms the foundation for the fractional understanding that the number of pieces into which a figure is divided is inversely related to the size of the pieces. Students have opportunities to apply this understanding throughout the development of both their fractional number sense and their intuition about the relative magnitude, or size, of fractions.

In grade 2, students also encounter experiences with geometric concepts that will help them develop an understanding of fractional parts. The composition and decomposition of two-dimensional shapes are some examples. These experiences can help students see how smaller shapes can be combined to create a larger shape and how a larger shape can be divided to create smaller shapes. Students can apply this understanding as they use pattern blocks and area models to represent fractional parts of a whole. Figure 3.6 shows a grade 2 example of how a square can be decomposed into four congruent triangles. To the right of the square is the same example used as a model to show fourths. Thus, students can use their grade 2 understanding of dividing a square into four triangles of the same size and shape as a basis for their grade 3 understanding that each triangle composes one fourth, or 1/4, of the square.

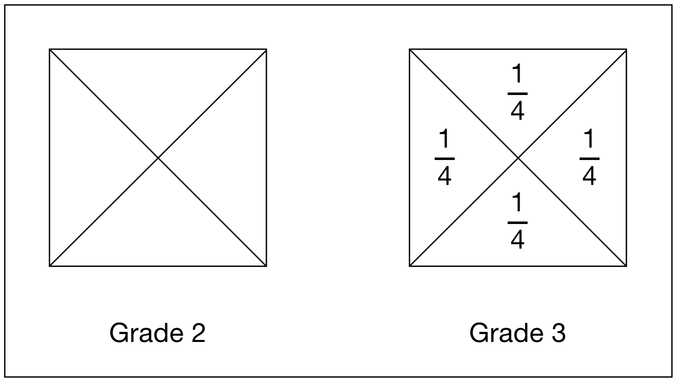


Fig. 3.6. Decomposition of a square and how it relates to fractions

Introducing and Building Fraction Concepts

In grade 3, students begin to focus on fractions. By looking at, and thinking about, applications of numbers in a variety of real-world contexts, students can be motivated to realize the need for numbers other than whole numbers. For example, when a bottle of juice is poured into identical cups, the juice may fill several cups completely but only a part of the last cup. Students find that a whole number of cups is not sufficient to describe "how much" juice was in the bottle.

In the study of fractions, a variety of concrete and pictorial representations should be used to give students the opportunity to explore different ways of interpreting fractions. Students should use many different forms of area models to represent a fraction as a part of a whole. They should be exposed to a variety of different types of groups of objects to explore a fraction as a part of a set. They should also use linear models, such as fraction bars and number lines, to explore a fraction as a distance or point on the number line.

When beginning the study of fractions with inexperienced students, such common unit fractions as 1/2, 1/3, 1/4, and 1/5—fractions that have a numerator of 1—become the building blocks for all fractions and are a good starting point. Unit fractions are formed by dividing a whole (whether the whole is a region, a set, or a distance on a number line) into equal parts. Once students understand common unit fractions, they can begin to understand that nonunit fractions are composed of unit fractions. For example, as students analyze the fraction model for 3/4, they begin to develop the understanding that 3/4 is actually 3 one-fourths, and thus is the sum 1/4 + 1/4 + 1/4. The general idea built from this specific example—that $2/n$ is two $1/n$ pieces, that $3/n$ is three $1/n$ pieces, and so forth—is the foundation on which models and questions should be designed for instruction about fractions.

Using area models to represent a fraction as a part of a whole

One model used to introduce students to fractions is the area model. The area model shows a whole that is divided into equal parts. This model is used to show fractional parts of a whole, such as fourths. Both models in figure 3.7 can be used to show fourths. Each whole is divided into four equal parts, so each part is one fourth.

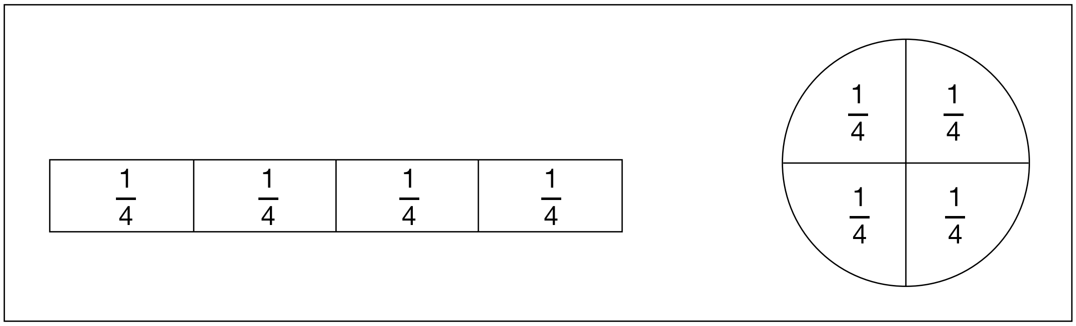


Fig. 3.7. Using area models to show fourths

In a slightly more sophisticated version of the fraction models in figure 3.7, parts of the whole are identified in some way; for example, parts are shaded. This version of the model is used to show a particular fraction. To interpret this model, students first identify the number of fractional parts in the whole. Then they count the number of parts that are identified. Both models in figure 3.8 can be used to show three fourths, or 3/4. To interpret these models, students observe that the whole is divided into four equal parts, so the models show fourths. Then they count the number of fourths that are shaded. Three of the fourths are shaded, so three fourths, or 3/4, of the region are shaded.

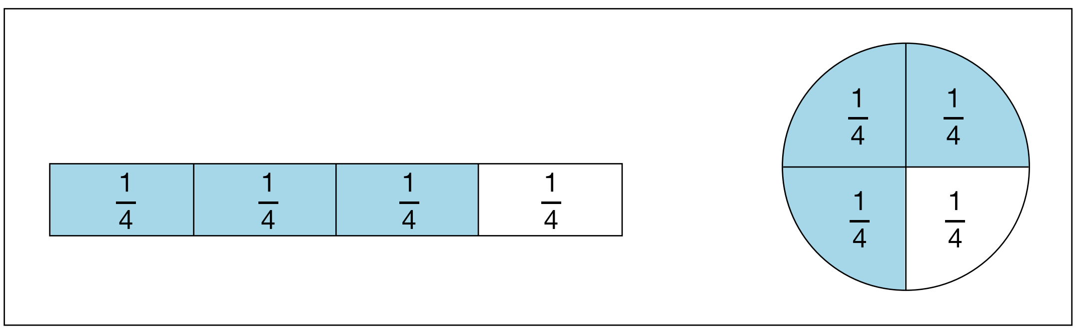


Fig. 3.8. Using area models to show 3/4

Through these models, students also begin to understand fractions that are equal to a whole, or 1. In the models in figure 3.8, if all four parts of the whole are shaded, then the whole is shaded. Students then observe that when considering fourths, 4 fourths, or 4/4, is equal to 1 whole. Through a variety of experiences with models, students can be led to the generalization that when given a whole that is divided into n parts (where n is greater than 0, of course) and all the fractional parts of the whole are considered, the model represents n/n, or 1 whole.

As mentioned previously, students must understand that an area model represents a fraction only if the parts into which the whole is divided are equal in size. Students can use the understanding of partitioning and equality developed in grade 2 to realize that in figure 3.9, the shape on the left has been partitioned into two equal-sized parts, and therefore the shading represents 1/2. The parts in the shape on the right, however, are not equal in size, so the shading does not represent the fraction 1/2.

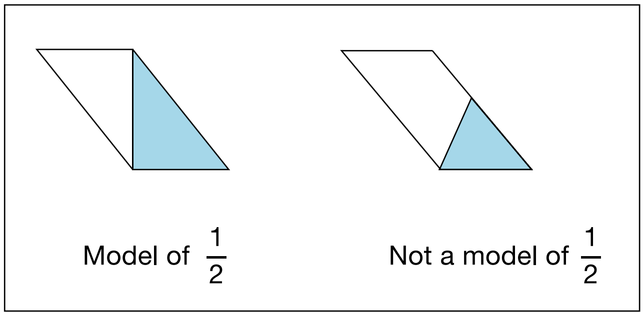


Fig. 3.9. Model representing 1/2 and model not representing 1/2

As students' exposure to area models increases, students can begin to draw their own area models to represent fractions. At this grade level, a drawing should simply illustrate a firm understanding of the meaning of equal parts of a whole, without showing parts that are exactly the same size. Note that in the example of students' drawings in figure 3.10, the partitions indicate an understanding of equal parts, even though the parts within each whole are not exactly equal.

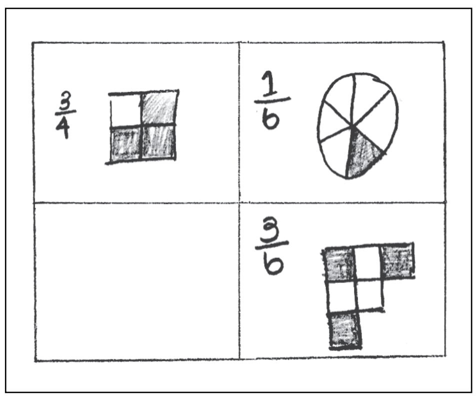


Fig. 3.10. Examples of area models drawn by students

Reprinted from *Navigating through Number and Operations in Grades 3–5* (Duncan et al. 2007, p. 31).

Using linear models to represent a fraction as a part of a whole

Linear models, such as fraction bars and the number line, can also be used to represent a fraction as a part of a whole. Fraction bars are linear models that help students make a natural transition from area models to the number line. The way in which fraction bars are placed end to end suggests a linear distance made up of equal sections between 0 and 1. However, the sections can also be interpreted as parts of a whole that can be taken apart and put together, much like area models of fractions. Fraction bars also can be labeled to make the parts evident to students.

Students can use concrete fraction bar manipulatives or fraction strips made of paper that can be folded. Fraction bars allow students to line up fractional pieces to show a fraction. For example, they can line up three 1/8 bars to show 3/8, as shown in figure 3.11.

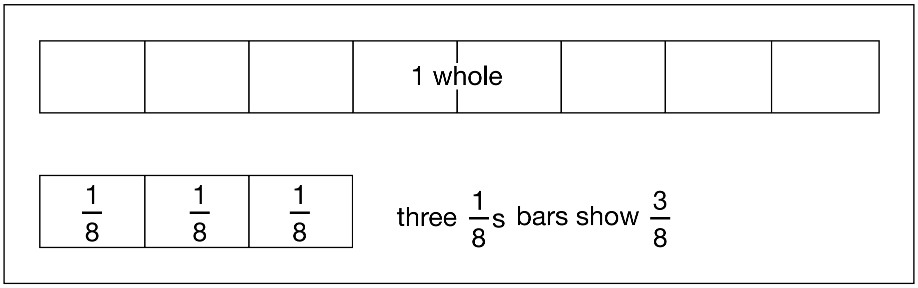


Fig 3.11. Using fraction bars to show $\frac{3}{8}$

Students can also fold strips of paper to emphasize the meaning of fractions. For example, if students fold one whole strip in half, they can see that the result is two equal sections and that each section is one half of the whole strip. If students fold the paper in half again, they can see that the result is now four equal sections and that each section is one fourth of the whole, and so on, as illustrated in figure 3.12.

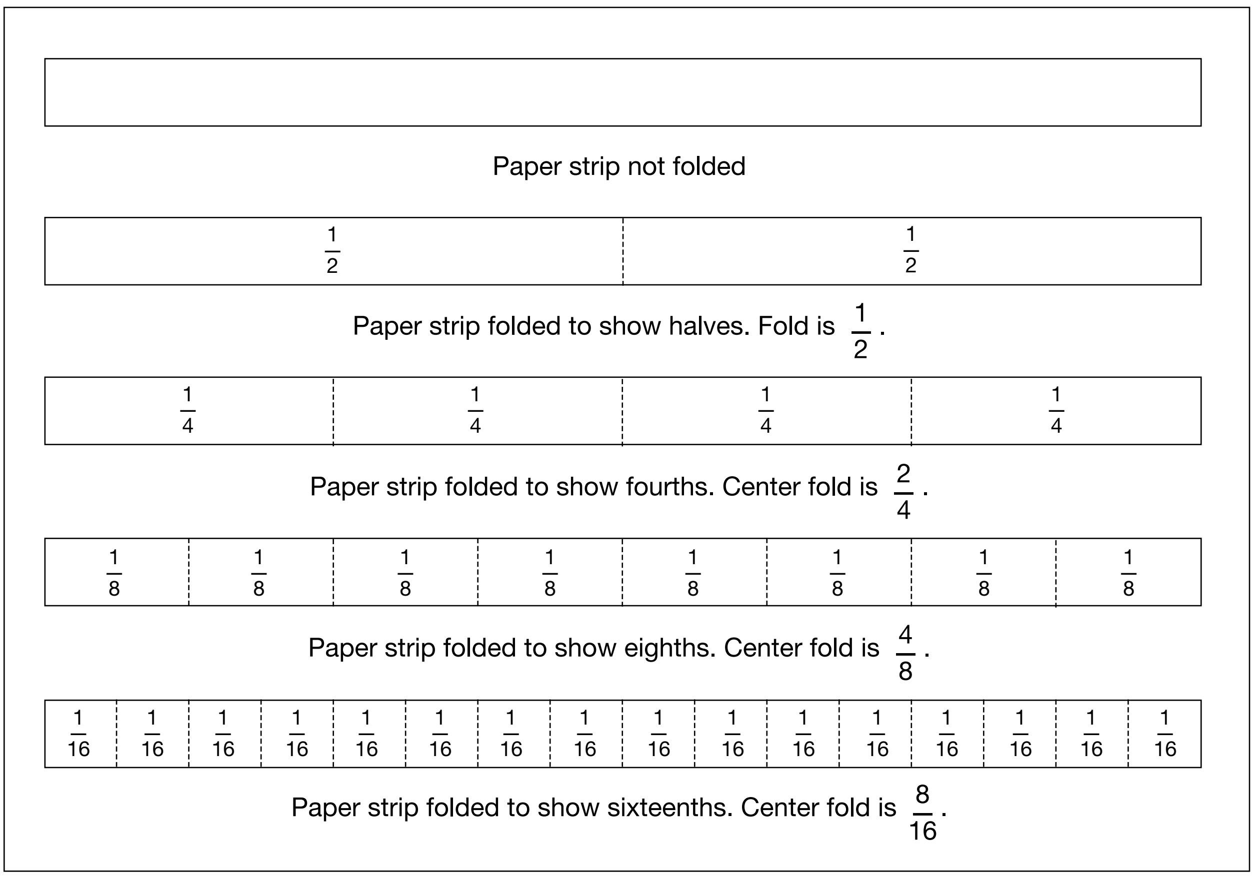


Fig. 3.12. Using paper folding to develop fraction concepts

In addition to modeling the basic meaning of a fraction as naming a part of a whole, these paper-folding experiences can illustrate other fundamental properties of fractions. Figure 3.12 shows that the section representing 1/2 is bigger than the section representing 1/4. If students continue to fold, they can see that the 1/8 section is even smaller than the 1/4 section, and so on. Thus, the crucial understanding of the inverse relationship between the number of parts in the whole and the size of the parts is again emphasized.

Through paper folding, students also begin to develop an intuitive sense of fractional equivalence. If they fold as illustrated in figure 3.12, they can see from their folded paper strip that the fold for 1/2 becomes the fold for 2/4, then for 4/8, and then for 8/16. This realization lays the conceptual groundwork that students will need as they begin to explore equivalent fractions. Finally, paper folding helps students begin to develop fractional number sense, or an intuition about how fractions are related to each other. For example, students see not only that 1/2 is greater than 1/4 but that 1/2 is double 1/4, because two 1/4 sections appear in 1/2 of the strip. As students investigate the relationships among the fractions modeled on the fraction strip, they begin to develop a sense of the relative magnitudes of fractions that they will apply in developing the concept of areas of fractions in grade 3, as well as later in grades 4 and 5.

Once students have had a number of experiences using fraction bars and fraction strips, they can begin to try to draw their own fraction bars. As mentioned previously with area models, a fraction-bar model should be considered "accurate" if students demonstrate their understanding of equal parts of a whole. Labels can clarify the drawing if the parts are not exactly the same size. Students can draw one whole bar first and then divide it equally to show fractional parts of a whole. They can also draw unit fraction bars to show the fraction. Both of these methods are reflected in figure 3.13.

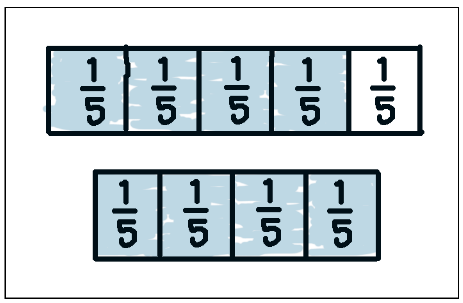


Fig. 3.13. Examples of fraction-bar models drawn by students

After students have a variety of experiences with fraction bars and fraction strips, they can move to a more sophisticated linear model, the number line. The number line serves to model fractions in several ways. First, the number line shows a distance divided into equal parts. So if 8 parts appear between 0 and 1 (see fig. 3.14), the first part is 1/8 of the distance to 1. In addition, the number line enables students to see fractions in relation to each other and helps develop an understanding of such benchmark numbers as 0, 1/2, and 1. If students observe, for example, that 3/8 is to the left of 3/4, they know from their work with whole numbers that 3/8 must be less than 3/4.

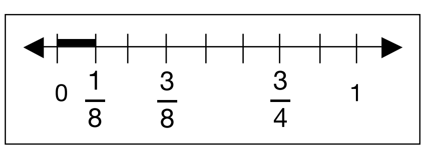


Fig. 3.14. Using a number line to show and compare fractions

In the same way that it is used in modeling whole numbers in earlier grades, the number-line model consists of a line that is divided by small vertical line segments that are labeled to show length from the zero point. The number-line model for fractions is a length model in which unit fractions are marked as equal lengths between 0 and 1 (as well as between other consecutive whole numbers). However, the lengths can get "lost" visually, and students often focus on the marks between the whole numbers rather than the length from 0 to a given mark (or the length between two unit-fraction marks). Students who have difficulty creating or reading a number line labeled with fractions should be guided to identify the length that is being used (the distance from 0 to the point labeled with the fraction) by drawing a bar or thin oval from the beginning of the length (0) to the end of the length (the point labeled with the fraction.) As with the number-line model for whole numbers, students need to focus on the lengths in the number-line model for fractions if they are to be able to use the number line to model operations with fractions in later grades.

Work with a number-line model can deepen students' understanding of the relative magnitudes of fractions. Also, fractional number-line models help students connect the familiar idea of linear measurement using a ruler with the less familiar concept of fractions. Students can visualize the number line as a ruler that measures the distance between two points. As students' understanding of the number line grows, they can begin to draw their own number-line models. Again, students' hand-drawn number lines may not show parts that are exactly the same size, but the drawings should illustrate students' understanding of equal parts of a whole. See figure 3.15.

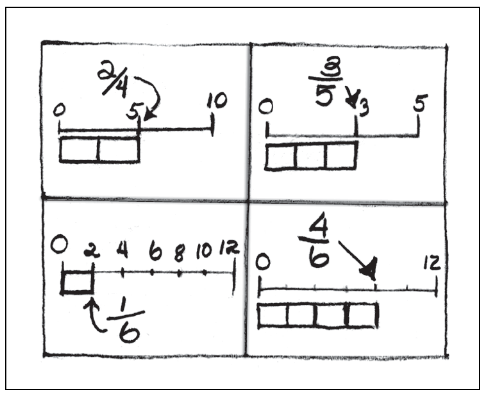


Fig. 3.15. Examples of number-line models drawn by students

Reprinted from *Navigating through Number and Operations in Grades 3–5* (Duncan et al. 2007, p. 31).

Understanding the size of fractional parts of a whole

Through a variety of representations or models, students begin to see fractions modeled in many different ways. For example, 1/8 can look like the area of a small rectangle in a larger rectangle, a portion of a circle, or a distance on the number line, as shown in figure 3.16.

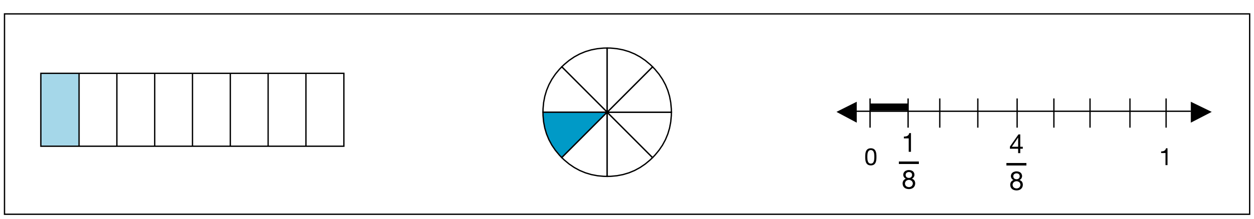


Fig. 3.16. Different ways to represent 1/8

These pictorial and concrete representations lay the foundation that students need to understand another important fraction concept: that the size of the fractional part is relative to the size of the whole. As is illustrated in figure 3.17, 1/2 of a large whole is not the same size as 1/2 of a smaller whole.

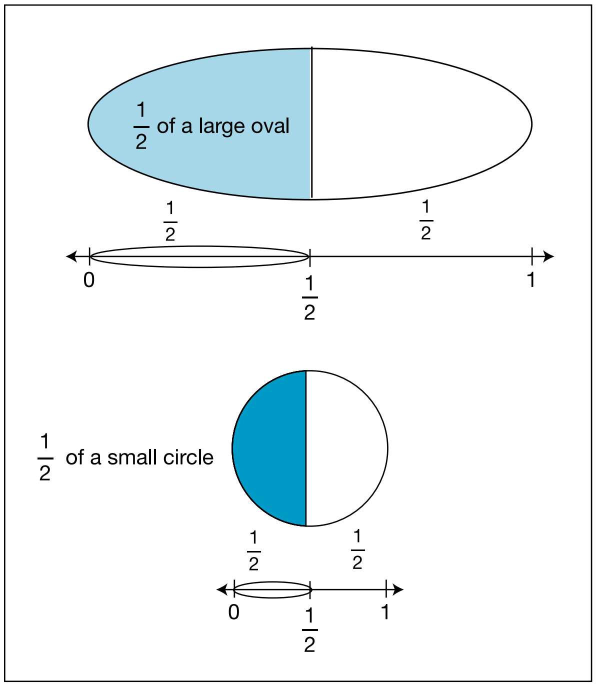


Fig. 3.17. Pictorial representation showing that 1/2 of a larger whole is more than 1/2 of a smaller whole

Another aspect in understanding the size of a fractional part is realizing that a given quantity can represent different fractions depending on the whole. This idea is important as students begin to apply their understanding of fractions to problem-solving situations. This relationship is illustrated in figure 3.18. For further ideas and information on exploring fraction ideas related to parts of wholes, refer to pages 27–33 in *Navigating through Number and Operations in Grades 3–5* (Duncan et al. 2007).

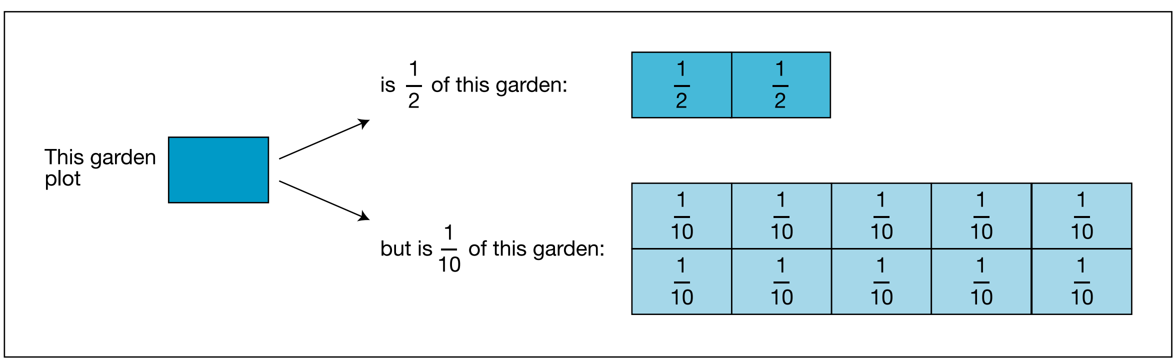


Fig. 3.18. Fractions depend on the whole: for example, a given quantity can be 1/2 of one whole but 1/10 of another whole.

Using groups of objects to model a fraction as a part of a set

When a fraction is used to name a part of a whole or a distance on a number line, the whole is a single contiguous unit that is divided into parts. However, a whole also can be represented as a set of discrete objects, and a fraction can be understood as describing a part of that set. When modeling a fraction as part of a set, the whole consists of the collection of objects that make up the set. A fraction is used to describe a certain part of the set. To understand using a set model to interpret a fraction, consider the collection of objects in figure 3.19. In this model, the whole is the set of six shapes. The "equal" parts in the model are the individual shapes. Because six shapes are involved, each shape is one sixth of the whole set. So, because four of the shapes are hearts, four sixths, or 4/6, of the set are hearts. Also, because two of the shapes are suns, the model can also be used to show two sixths, or 2/6.

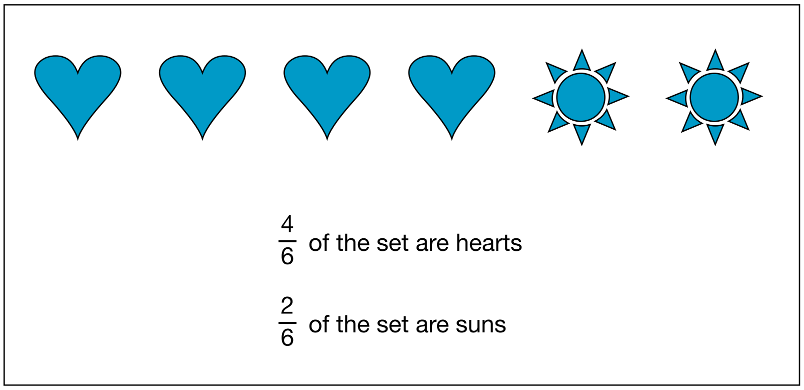


Fig. 3.19. Representing a fraction as a part of a set

Students should note that in the example in figure 3.19, the number of hearts is not the same as the number of suns, and that all the shapes are shaded. Students should be exposed to a wide variety of set models that are pictorial as well as concrete. This variation in examples will encourage students to develop flexibility and fluency in their understanding of fractions of a set. As students use sets to model fractions, they have yet another opportunity to understand the concept of the fraction that represents one whole. For example, in figure 3.19, all the shapes are shaded, so the model also illustrates that six sixths, or 6/6, of the set are shaded. Because 6/6 represents the whole set, 6/6 = 1.

As students mature in their understanding of fractional parts of sets, they can begin to create their own set models. Students should be encouraged to use drawings as well as real objects, including people, to model fractions. For example, if students have five friends and three of them are wearing sneakers, they can say that 3/5 of their friends are wearing sneakers. In this example, the group of five friends is the whole set. Each friend is 1/5 of the set, and because three of them are wearing sneakers, three fifths, or 3/5, of the members of the set are wearing sneakers. Because the objects in a set do not have to be the same size or shape, any set that correctly models a fraction can be used. See figure 3.20 for examples of students' set drawings.

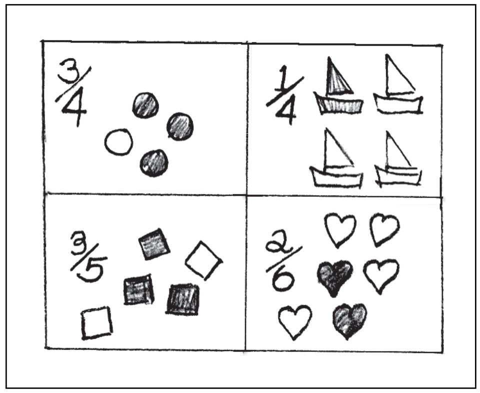


Fig. 3.20. Examples of set models drawn by students

Reprinted from *Navigating through Number and Operations in Grades 3–5* (Duncan et al. 2007, p. 31).

Using language to develop fraction fluency

Using correct mathematical language with students as they begin their exploration of fractions is just as important as using accurate fraction models. Students need to translate the understanding that they develop through their work with concrete and pictorial models to abstract symbolic representations for fractions. Students need to possess the ability to write fractions accurately as well as the ability to understand the symbolic form of a fraction if they are to progress to more complicated fraction concepts, such as decimal-fractional equivalence and fraction computation.

In grade 3, students work toward understanding the meaning of a fraction as it is written and spoken. Through a variety of concrete experiences as well as interpretation of written fractions, students begin to understand that a fraction consists of two parts: the numerator, or top number, and the denominator, or bottom number. The numerator of a fraction tells the number of parts. The denominator of a fraction tells how many equal parts are in the whole by naming the type of parts being used (e.g., fourths, fifths, sixths, etc.). This naming makes sense because the word *numerator* comes from the word *number* and the word *denominator* comes from the word *name*. As students interpret the different fraction models, they learn that to write the fraction represented by the model, they need to determine the number of equal parts in the whole (which identifies the name of the parts) and write this number as the denominator. Then they need to determine the number of parts being described and write this number as the numerator. In figure 3.21, the area model shows a region divided into six equal parts. The set model shows six shapes, each of which counts as one part of the whole. In each model, the whole has six equal parts, and four parts are shaded. In both models, the name of the parts is sixths and the number of parts being considered is four. The fraction 4/6 can be used to describe the designated portion in both models.

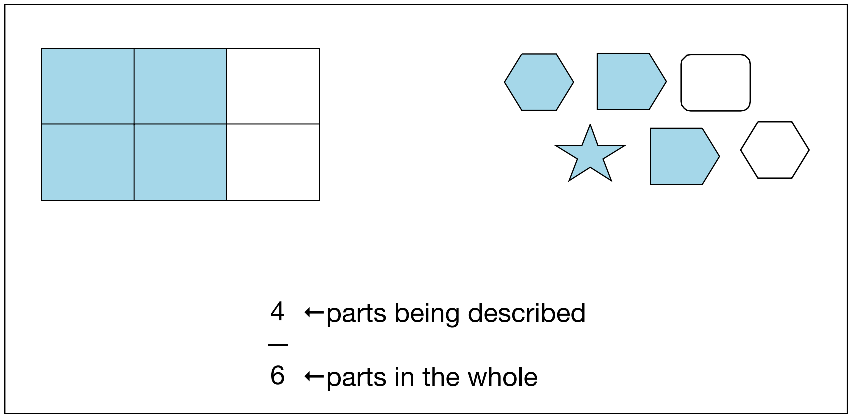


Fig. 3.21. Area and set models that show how to write a fraction

Looking again at figure 3.21, notice that when the fraction "four-sixths" is read, the number of parts being described is said first, "four." Then the name of the parts in the whole is said, "sixths." Note that the number of parts in the whole indicates the name of the parts. Six parts are involved, so each part is one sixth of the whole. In other languages, fractions are read differently. For example, in some languages, the fraction 4/6 is read as "out of six parts, take four," so that the number of the parts in the whole is indicated first and then the number of parts is stated. Notice, too, that the English vocabulary words that are associated with the number of parts in the whole—for example, *third, fourth, fifth, sixth,* and so on—are the same as the ordinal words we use to indicate position, such as in "third in line," "fourth in line," and so on. Teaching students to use more meaningful language, such as saying 4/6 as "out of six parts, take four," in conjunction with the English fraction term, "four-sixths," may help students develop an initial understanding of fractions.

Representing fractions that are equal to, less than, or greater than 1

Students begin their exploration of fractions by modeling and learning about fractions between 0 and 1. A solid conceptual understanding of fractions less than 1 is pivotal to broadening students' understanding to include fractions greater than or equal to 1. Any discussion of fractions greater than 1 must also include fractions equal to 1. This concept is fairly intuitive for students who have had many opportunities to explore area, set, and number-line models that have already included representations of one whole. The models in figure 3.22 link the concept of one whole with the types of models that should already be familiar to students.

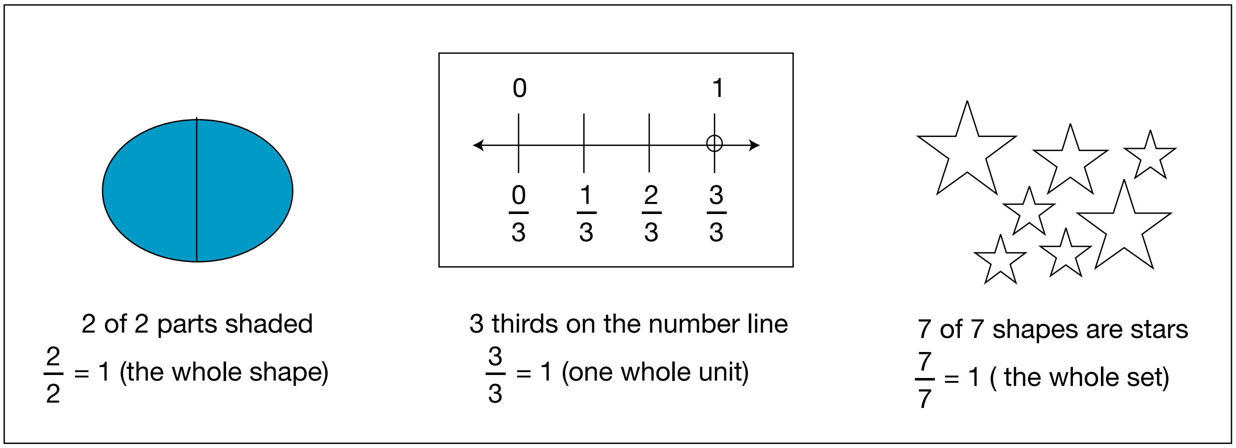


Fig. 3.22. Examples of fractions equal to 1

As students mature in their understanding of fractions, they integrate this concept of "all parts" with fractional symbols and language. Thus, students can be led to observe that when all the parts of a whole are taken, the numerator and denominator are the same number and the value of the fraction is equal to 1.

Once students understand fractions less than 1 and fractions equal to 1, they can move on to explore fractions greater than 1. Again, fraction models with which students are familiar can help students make the transition from their understanding of fractions less than 1 to fractions greater than 1. In figure 3.23, the area model shows that the oval is one whole and it is divided into two parts, so each part represents 1/2. When another oval, identical to the first, is added, the model begins to represent a value greater than 1. Each half of an oval represents one half. To find the number of halves that are represented in the model, students count halves. They count 3 halves, so the model shows 3/2. Students can understand from the model that since more than one oval is shown, the fraction 3/2 must be greater than one whole.

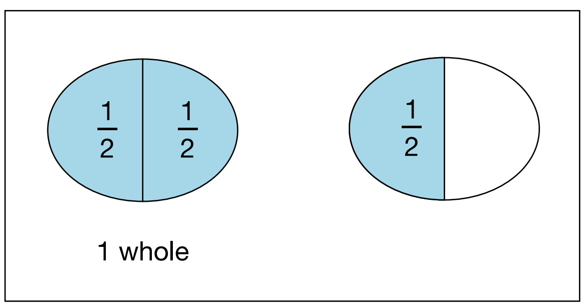


Fig. 3.23. Area model showing that 3/2 is greater than 1 whole

Students can also use a linear model to show fractions greater than 1. Using the linear model, students can continue to count fractional parts past one whole. Students have used the model in the example in figure 3.24 to count fourths: 1/4, 2/4, 3/4, and 4/4. With this model, they can continue counting fourths past 4/4, or one whole: 5/4, 6/4, 7/4, and so on. As students' understanding of fractions greater than 1 deepens, they begin to realize that when a fraction is greater than 1, the numerator is greater than the denominator. Having a variety of opportunities to explore fractions greater than 1 through concrete and pictorial models can help students understand that in a fraction greater than 1, the numerator is greater than the denominator because the number of parts being described is greater than the number of parts in one whole. Fractions greater than or equal to 1, in which the numerator is greater than or equal to the denominator, are called *improper fractions*.

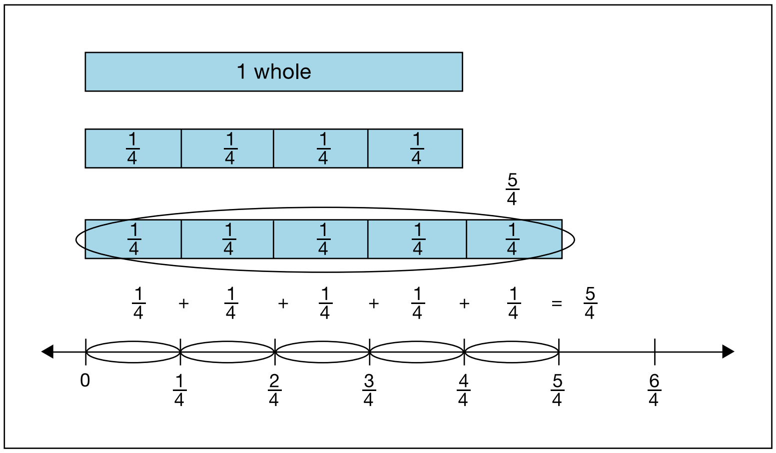


Fig. 3.24. Linear models showing the improper fraction 5/4

Models used to help students understand and write fractions greater than 1 can also help students understand the equivalence between fractions greater than 1 and mixed numbers. A mixed number is the sum of a whole number and a fraction. The model of a value greater than 1 can be interpreted both as a model of an improper fraction and as a model of a mixed number. Students first apply their understanding of fractions to interpret the model. For example, in figure 3.25, students can see that one whole is divided into three equal parts, so each part is 1 third. Then they can count thirds to find the name of the fraction represented. Since seven thirds are shown, the model represents 7/3. Students can then interpret this same model with respect to whole numbers and fractional parts of whole numbers. For example, they see that two wholes are shaded. They can also see that 1/3 of another whole is shaded. Thus, the model shows that two wholes and 1/3 of a whole are shaded, which can be represented by $2 + \frac{1}{3}$, or $2\frac{1}{3}$. By interpreting the same model in these two ways, students can naturally establish the equivalence between $\frac{7}{3}$ and $2\frac{1}{3}$. Through concrete and pictorial examples such as these, students can be led to the conclusion that an amount greater than 1 can be written as an improper fraction as well as an equivalent mixed number.

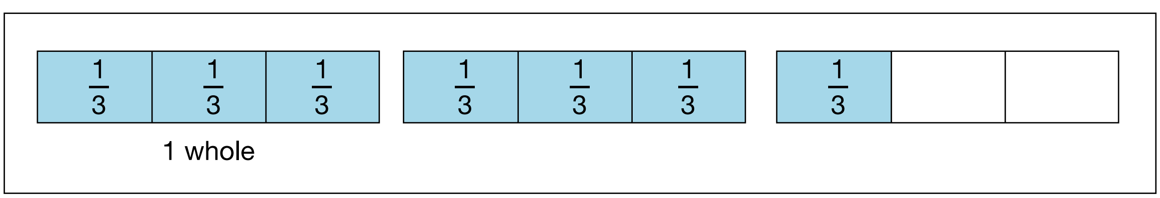


Fig. 3.25. Using a model to establish the equivalence between 7/3 and $2\frac{1}{3}$

Equivalent fractions

As students work with concrete materials and pictorial representations, they can begin to develop a sense that different fractions can be used to represent the same portion of a whole. For example, when they fold paper, they can be asked to notice that 1/2, 2/4, and 4/8 are all represented by the same fold line. Also, as they work with area models, their attention can be drawn to the fact that 1/3, 2/6, and 3/9 all represent the same part of the rectangle, as shown in figure 3.26.

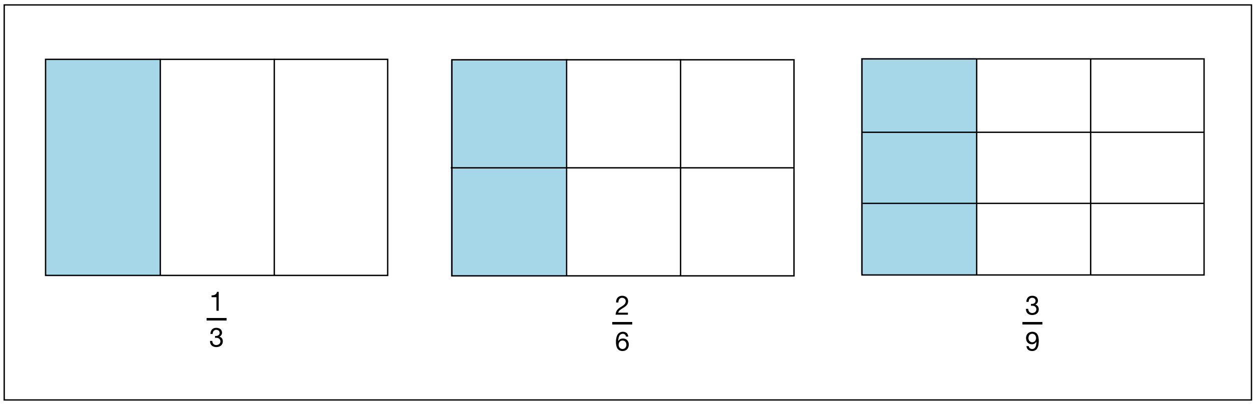


Fig. 3.26. Area models representing equivalent fractions

Purposeful experiences with the same models that students have used to model fractions can make the discussion of equivalent fractions an intuitive extension of previous fraction concept development. As shown in the linear models in figure 3.27, students can visualize that equivalent fractions represent the same portion of a whole. In the model, 2/3, 4/6, 6/9, and 8/12 all show the same portion of the bar. Thus, students learn that 2/3 = 4/6 = 6/9 = 8/12.

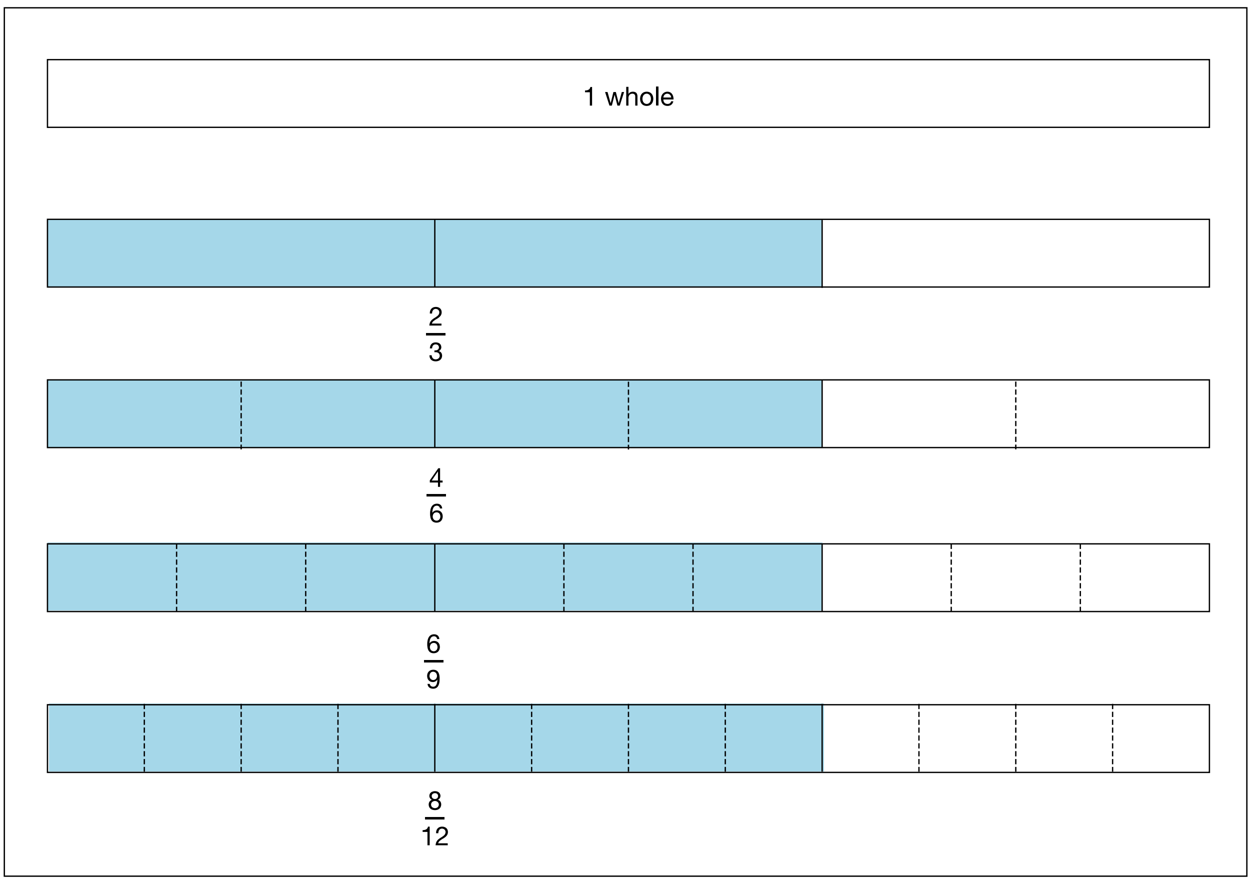


Fig. 3.27. Linear model showing that equivalent fractions name the same part of the whole

Using models such as these again emphasizes the inverse relationship between the number of parts in the whole and the size of each part. Students can internalize from the models in figures 3.26 and 3.27 that the more pieces into which the whole is divided, the smaller the size of each of the pieces and the more pieces that are needed to make up the same amount. The value of emphasizing this important relationship throughout the development of fraction concepts will become apparent as students move into comparing and ordering fractions.

Another pictorial representation that aids students in their understanding of equivalent fractions is the number line. Figure 3.28 pictures multiple number-line models showing equivalent fractions for 1/2. Some students benefit from seeing all these number lines superimposed on one another.

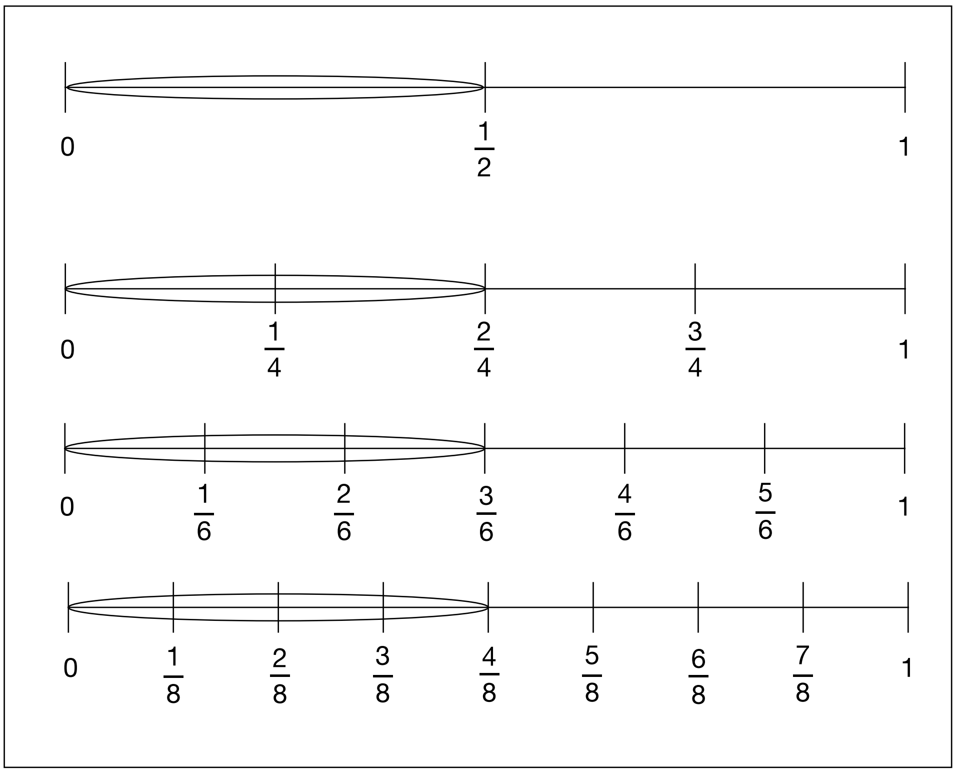


Fig. 3.28. Number-line models showing equivalent fractions for 1/2

As in the paper-folding example, students can see that 1/2, 2/4, 3/6, and 4/8 are located at the same point on the number line. That is, they are all the same distance from 0 and the same distance from 1, so they are equivalent fractions.

Comparing and ordering fractions

When students compare two fractions that are not equal, they determine which fraction is greater.

Comparing unit fractions

The meaningful study of the magnitude of fractions begins with the foundational principle of the inverse relationship between the number of equal parts in a whole and the size of the parts, specifically, that as the number of equal pieces in a whole increases, the size of the fractional pieces decreases. This idea is not unrelated to the idea in linear measurement developed in grade 2 that the smaller the unit used to measure a given length, the larger the number of units needed for the measurement. The fractional relationship is shown in figure 3.29.

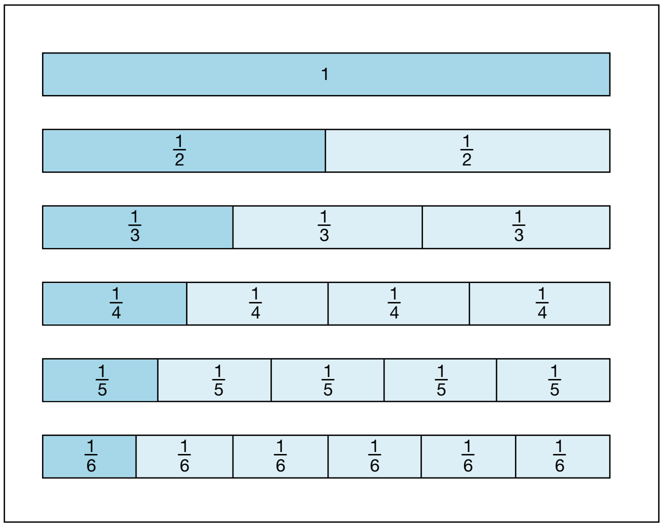


Fig. 3.29. Pictorial representation of the idea that the greater the denominator, the smaller the size of the unit fraction

If students have explored this concept informally throughout the development of other fraction concepts, its formalization in the study of comparing fractions will be a natural extension of previous work. Previous pictorial and concrete experiences with fractions are crucial in the development of this concept because, in their work with whole numbers, students need to have learned that, for example, 4 is greater than 3. In working with fractions, however, they must understand that, with regard to the same whole, fourths are smaller than thirds. They can then apply this concept in a meaningful way to compare fourths and thirds. As a result of many opportunities to model fractions, students begin to understand that, with regard to the same whole, the greater the denominator, the smaller the unit fraction.

Real-life situations can also help illustrate this concept for students. For example, students may intuitively know the answer to the question "Would you rather have 1/4 or 1/5 of the cake?" Students use their basic understanding of fractions to realize that they would prefer 1/4 of the cake because it is bigger than 1/5 of the cake. Students who understand the fundamentals of fractions know that they get 1/4 of the cake when the cake is shared among four people, and they get 1/5 of the cake when the cake is shared among five people, and that sharing a cake among four people gives each person more cake than sharing the cake among five people.

Comparing fractions that have the same denominator

Another type of comparison that is fairly easy for students to make is between fractions having the same denominator and different numerators. Much of the previous work in fractions has given students the experience to understand that when fractions have the same denominator, the wholes have been divided into the same number of equal parts, so the fractional parts that would be represented by a unit fraction are the same size. As is illustrated in figure 3.30, the fraction with the larger numerator has the larger number of equal parts, so it describes a larger portion of the given whole.

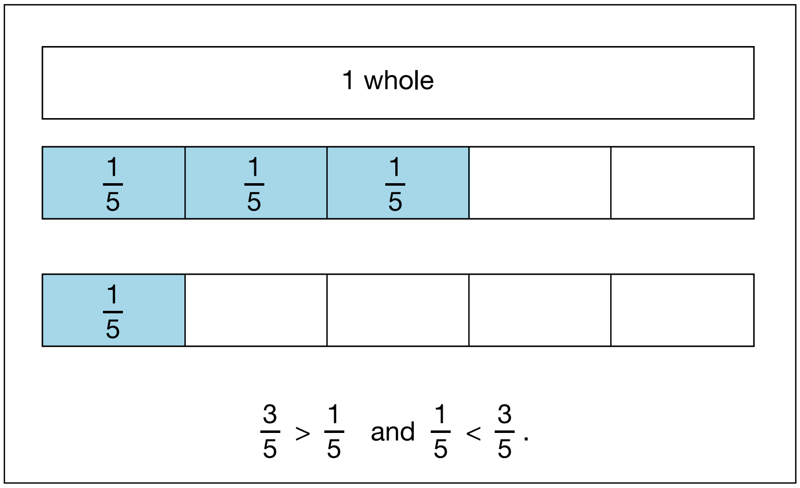


Fig. 3.30. Pictorial representation of the principle that if the denominators are the same, the fraction with the larger numerator represents the larger fraction

Comparing fractions that have the same numerator

To compare fractions that have the same numerator and different denominators, students must revisit the role of the denominator in comparing unit fractions. Using the same principles that they applied to comparing unit fractions, students can extend their understanding of the magnitudes of other fractions that have the same numerator. Figure 3.31 presents models for 3/8 and 3/10. Students can see in the models that, for the same whole, tenths are smaller than eighths. Students can then infer that the same number of smaller pieces is less than the same number of bigger pieces, thus 3/10 is less than 3/8. After many concrete and pictorial experiences with these kinds of fractions, students can begin to generalize that if the numerators are the same, the fraction with the larger denominator describes the smaller portion of the given whole.

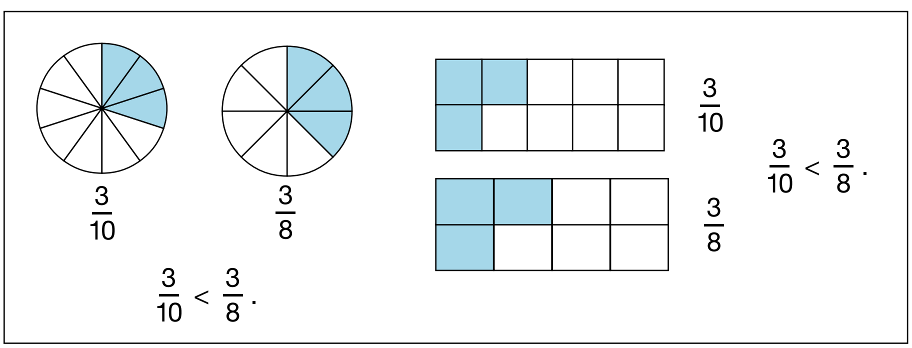


Fig. 3.31. Pictorial representation of the principle that if the numerators are the same, the fraction with the larger denominator represents the smaller fraction

Using benchmarks to compare fractions

As students mature in their ability to compare numbers, they will sometimes find that a more efficient way to compare fractions is by considering how they relate to benchmark numbers, such as 0, 1/2, or 1. As students relate fractions to benchmark numbers, they continue to develop their fractional number sense.

Pictorial and concrete models, as well as the number sense and reasoning that they have been developing, can help students compare fractions by using benchmarks. For example, if a student knows that one fraction is less than 1/2 and another fraction is greater than 1/2, then the student can apply the reasoning learned in grade 2 to determine that the first fraction must be less than the second fraction. The model in figure 3.32 also illustrates this concept.

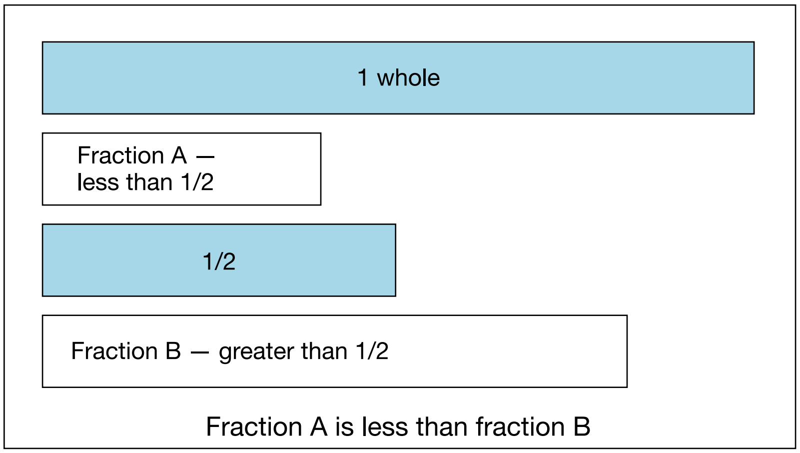


Fig. 3.32. Pictorial representation of the principle that if fraction A is less than 1/2 and fraction B is greater than 1/2, then fraction A is less than fraction B

Students can also compare fractions with the benchmark number 1. For example in figure 3.33, to compare 4/5 and 7/8, students can relate each fraction to 1. Consider this reasoning: 4/5 is 1/5 away from 1, whereas 7/8 is 1/8 away from 1; 1/8 is less than 1/5, so 7/8 is closer to 1 than 4/5 is; thus 7/8 > 4/5. As is evident from this reasoning, the relationship between the size of the parts and the number of parts in a whole again plays a crucial role in understanding how to compare fractions.

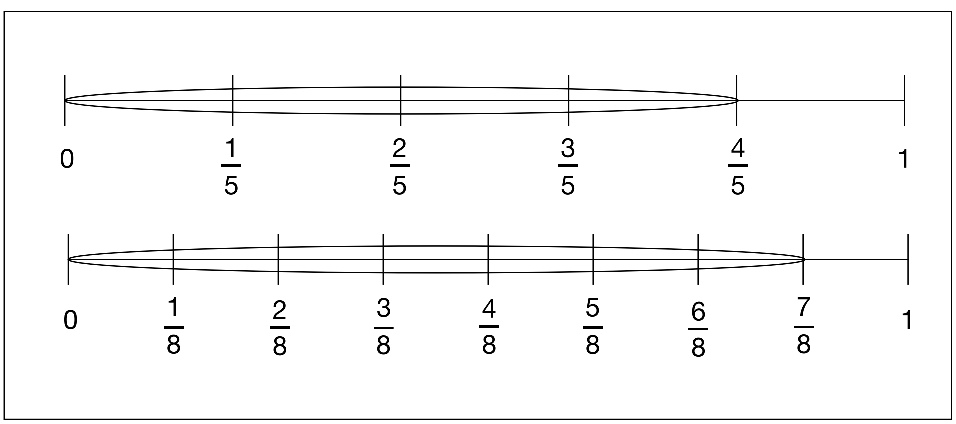


Fig. 3.33. Pictorial representation comparing fractions by using 1 as a benchmark

Ordering fractions

Many concrete and pictorial experiences using such models as the number line can enhance students' ability to use reasoning and benchmark numbers to order fractions. The number line is a particularly powerful tool for ordering fractions in grade 3 because students are already familiar with its sequential left-to-right progression of numbers from their work with whole numbers in grade 2. Students can also apply ideas about the relative magnitudes of counting numbers that they developed in grade 2 and that allow them to order, for example, 3, 7, and 10. As students' ability to synthesize these previously learned concepts matures, they will be able to apply them to ordering fractions. For example, they can use a mental image of the number line, as in figure 3.34, along with the following reasoning to order 1/4, 1/5, and 5/8: 1/5 is less than 1/4, and 1/4 is less than 1/2, and 5/8 is greater than 1/2; so the order of the numbers from least to greatest must be 1/5, 1/4, 5/8. A good foundation in understanding whole numbers coupled with a thorough understanding of the meaning of fractions and how to use benchmarks to compare them facilitates students' ability to order fractions.

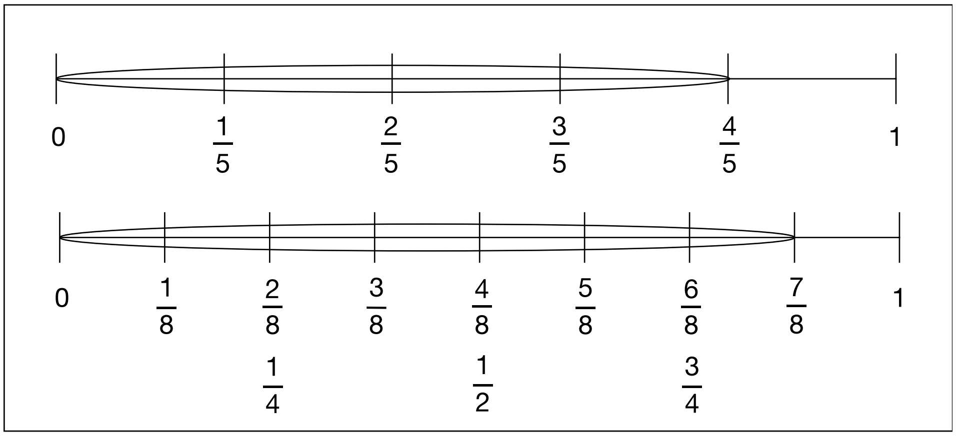


Fig. 3.34. Using the number line and benchmark numbers to order fractions

Strengthening Fluency through Connections

Students must be exposed to and understand connections among mathematics concepts. Such connections lead to a deeper understanding of mathematics and how concepts build on one another and are related both within a grade level and across grade levels. Connections with other mathematics concepts at this grade level are evident throughout this Focal Point.

In grade 3, the connection between fractions and measurement should be highlighted. As previously mentioned, relating the number line to the ruler enables students to better understand the number line as a fraction model. These linear models, the ruler and the number line, work together to enhance students' understanding of both fractions and measurement. When working with customary units (e.g., inches), students begin by using a ruler to measure length to the nearest whole unit and then move to measuring lengths to the nearest 1/2 and 1/4 unit. The ruler gives students a concrete way to conceptualize that if, for example, a length extends about halfway between 0 and 1 inch, the length is about 1/2 inch long. This understanding lays the foundation for students to understand how the number line can be used to model fractions. Then, as students become familiar with the number line as a fraction model in grade 3, they continue to use it to model fractions with larger denominators, including, for example, eighths and sixteenths. Students then can use this deepening understanding of the number line to translate back to measurement as they move on to using a ruler to measure lengths with greater and greater precision, including, for example, to the nearest eighth and sixteenth of an inch.

Connections in later grades

As grade 3 students learn about the meaning of fractions and fraction equivalence, they are laying the foundation that will enable them in grade 4 to connect fractions and decimals. For example, a grade 3 student may see the visual representation in figure 3.35 and interpret it as an area model that shows 1/2 because one of the two equal parts is shaded. As students' knowledge of fractions matures, the model can be interpreted by students as 1/2 = 50/100. Students will then have the foundation needed in grade 4 to connect hundredths as a denominator of a fraction with hundredths as a decimal place value. If they have a deep understanding of fractions and equivalent fractions, they will be able to conclude that 1/2 = 50/100 = 0.50.

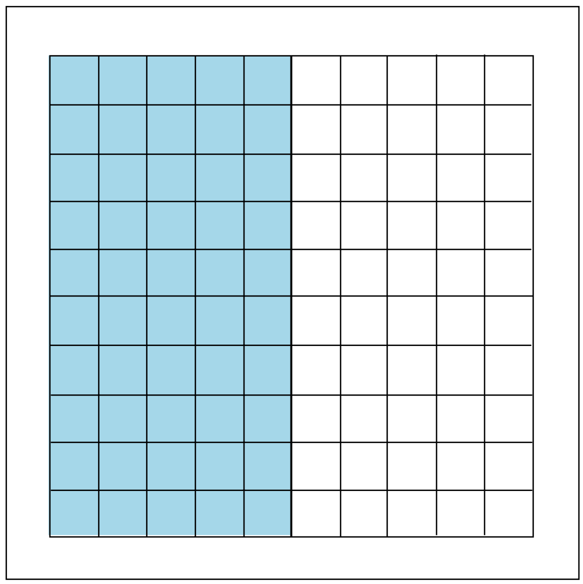


Fig. 3.35. Pictorial example demonstrating that 1/2 = 50/100 = 0.50

Students also use their fractional number sense developed in grade 3 to learn how to add and subtract fractions in grade 5. The groundwork for this computation was laid when students realized from observing the various models that 3/5 is three 1/5 pieces (or 3/5 = 1/5 + 1/5 + 1/5) and that joining a 1/5 piece and a 2/5 piece results in a 3/5 piece (or 1/5 + 2/5 = 3/5).

As students develop a firm grasp of the meaning of fractions, the foundations for understanding the connection between division and fractions are also being formed. As students use correct mathematical language to describe fraction models showing, for example, 1/4 as one whole divided into four equal parts, they can begin to develop the intuitive sense that 1/4 is a representation of 1 divided by 4, or 1 ÷ 4. This connection will be important as they learn to understand the division of fractions in grades 4 and 5 and beyond.

Building Depth of Understanding

The concept of fractions and fraction equivalence can be very abstract for young students. After reading about focusing on fractions as suggested in the second Focal Point for grade 3, consider how you might make fractions more meaningful for students. How do you think this approach will help increase your students' depth of understanding?

4 Focusing on Two-Dimensional Shapes

In grade 3, students continue to develop their understanding of two-dimensional shapes by building on what they learned about shapes in prekindergarten through grade 2. The focus in grade 3 is on building the visualization and visual discrimination skills that enable students to recognize types of shapes. Students will apply these visualization skills to decompose, combine, and compare shapes and to recognize congruence and symmetry.

Instructional Progression for Two-Dimensional Shapes

The focus on two-dimensional shapes in grade 3 is supported by a progression of related mathematical ideas before and after grade 3, as shown in table 4.1. To understand what students need to know to do well in grade 3, we briefly discuss some of the important ideas that students focus on in prekindergarten through grade 2 that prepare them for describing, analyzing, and classifying two-dimensional shapes. We also discuss how students will use this learning from grade 3 to find the areas of two-dimensional shapes including polygons in grade 4. This introduction to area in grade 4 builds on students' prior knowledge of decomposing shapes to find the area of polygons and helps them understand the formula for finding the area of a rectangle. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books, for example, *Focus in Grade 2* and *Focus in Grade 4*.

Table 4.1 represents an instructional progression for developing understanding of two-dimensional shapes in kindergarten through grade 4.

Early Foundations in Geometry

When young children begin school, they already possess many rudimentary concepts of shape and space that form the foundation of the geometric knowledge and spatial reasoning that they develop throughout their school years (*Principles and Standards for School Mathematics* [NCTM 2000]). In the early grades, this knowledge is expanded, investigated, and discussed. This foundation will enable students to build on their own personal experiences to create a solid conceptual understanding of two- and three-dimensional shapes.

In the early development of geometry concepts, students begin recognizing shapes by their general appearance and describing them in terms of familiar objects, such as a block or a ball. Through a variety of experiences, students learn the defining properties of such shapes as squares, triangles, circles, and rectangles, as well as how to recognize that shapes in the same category may vary by shape, size, or orientation. For example, the pictures in figure 4.1 can be used as a springboard for discussion about which in fact show triangles and which do not, and why. As students' ideas of "triangle" mature, they begin to understand that the triangles in figure 4.1(a) are all closed shapes with straight sides and three corners, whereas each figure in figure 4.1(b) is not a triangle because it is open, has curved sides, or has more than three corners. Discussions such as these enable students to move from describing shapes using their own informal geometric vocabulary to using the formal mathematical vocabulary for characteristics and properties of shapes.

Table 4.1
Grade 3: Focusing on Two-Dimensional Shapes—Instructional Progression for Developing Understanding in Kindergarten–Grade 4

Kindergarten–Grade 2	Grade 3	Grade 4
Students identify, name, and describe a variety of shapes, such as squares, triangles, circles, rectangles, hexagons, and trapezoids, presented in a variety of ways. (Kindergarten) Students use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes. (Kindergarten) Students compose and decompose two-dimensional shapes to understand the properties of the original and composite shapes. (Grade 1) Students recognize shapes in different orientations and perspectives, describe their attributes, and identify how they are alike and different. (Grade 1) Students compose and decompose two-dimensional shapes to develop a foundation for understanding area.*	Students build, draw, and analyze two-dimensional shapes to explore their attributes and properties. Students describe, analyze, compare, and classify two-dimensional shapes by their sides and angles. Students decompose, combine, and transform polygons to make other polygons to build foundations for area and fraction models. Students solve problems involving congruence and symmetry.	Students recognize area as an attribute of two-dimensional regions. Students learn that they can quantify area by finding the total number of same-sized units of area that cover a shape without gaps or overlaps. Students understand that a square that is 1 unit on a side is the standard unit for measuring area. Students select appropriate units, strategies (e.g, decomposing shapes), and tools to solve problems involving estimating or measuring area. Students make the connection between area and the area model that they have used to represent multiplication to justify the formula for the area of a rectangle.

*Appears in the Grade 2 Connections to the Focal Points

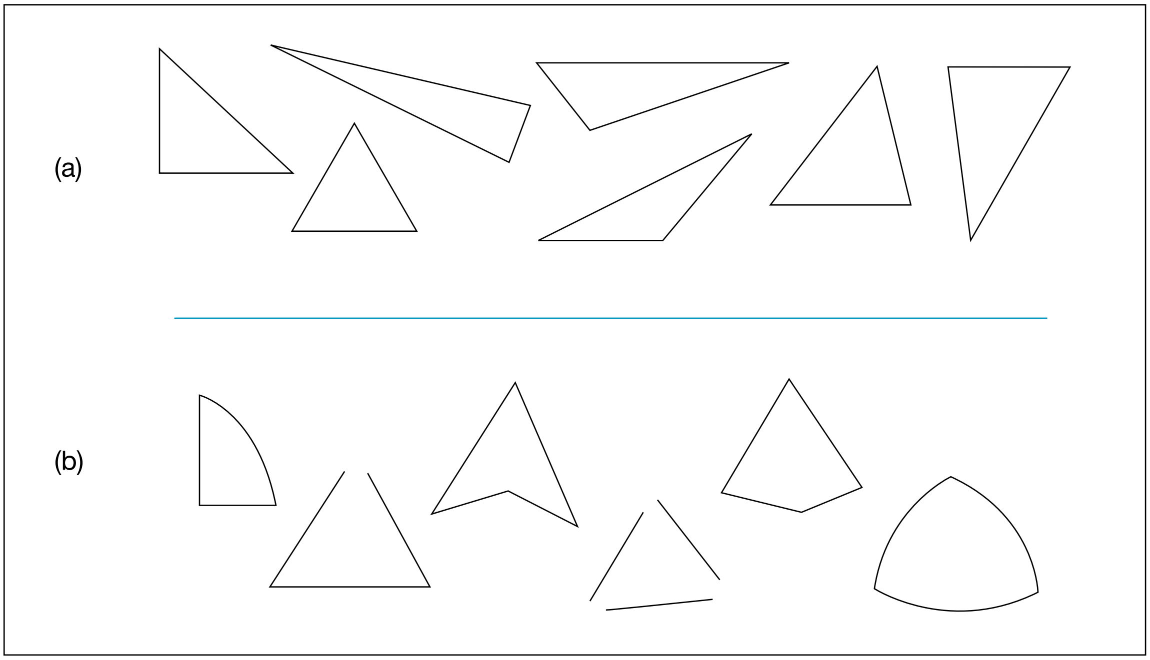


Fig. 4.1. Figures that highlight the defining characteristics of a triangle

Reprinted from *Principles and Standards for School Mathematics* (NCTM 2000, p. 98).

Students also begin to understand that orientation has no effect on the type of shape. Through many experiences with identifying shapes in a variety of orientations, students begin to realize that shapes can be the same, regardless of their position. Students will use this important conceptualization later in work with congruence and transformations.

Even in early explorations of geometric properties, students are introduced to the embeddedness of the categories of geometric shapes. For example, they learn that a square is a "special kind of rectangle" in which all sides have the same length. Early and frequent exposure to identifying a square as a rectangle and including squares as examples when discussing rectangles can help prevent students from developing the common misconception that a square is not a rectangle. These informal experiences are formalized later when students learn the definition of a rectangle as a quadrilateral with four right angles. Students who have been exposed to identifying a square as a type of rectangle will be able to use this formal definition coupled with their intuitive sense of the properties of a square later to make sense of such statements as "All squares are rectangles, but not all rectangles are squares."

In conjunction with identifying and naming shapes, students must also describe shapes on the basis of their attributes and properties. For example, inexperienced students may state that a triangle has "three points" or "three sides" without knowing what a "side" or "point" is (Clements 1987). Opportunities to work with concrete and pictorial representations, as well as technology, guide students to understand that the "sides" of a shape are the straight line segments that form the shape and that a "point" or "corner" is where the sides come together, or intersect, at a vertex. Students learn that a shape's sides and vertices can be used to describe, identify, and name the shape. This knowledge gives students the vocabulary necessary to say that, for example, a shape with exactly three sides that meet only at the ends to form exactly three corners (or vertices) and that enclose a space is a triangle, and a shape with exactly four sides that meet only at the ends to form exactly four corners (or vertices) and that enclose a space is a quadrilateral, and so on.

As students mature in their understanding of two-dimensional shapes, they can begin to use basic shape recognition and spatial reasoning to identify and manipulate geometric shapes in their environment. Students begin this process very early in their development when they use primary shape-sorting games, such as puzzles, in which they have to fit a certain shape in a certain orientation into a cutout or slot. To be successful in these experiences, students first have to be able to match one shape with another—for example, the shape of the puzzle cutout with the shape of the puzzle piece. Then they have to manipulate the shape so that it is oriented the same way as the cut-out shape. This type of activity is an example of even very young children's ability to apply ideas of shape and orientation to a real-world experience.

In grade 1, students build on their understanding of two-dimensional shapes as they combine and decompose two-dimensional shapes. When students combine shapes, they put together shapes to make a more complex shape. Some of these newly created shapes might be composed of pieces that are all the same shape, as shown in figure 4.2(a), or pieces of various shapes, as shown in figure 4.2(b). Students might even notice that they can create the same final shape with different combinations of shapes, as shown in figure 4.3.

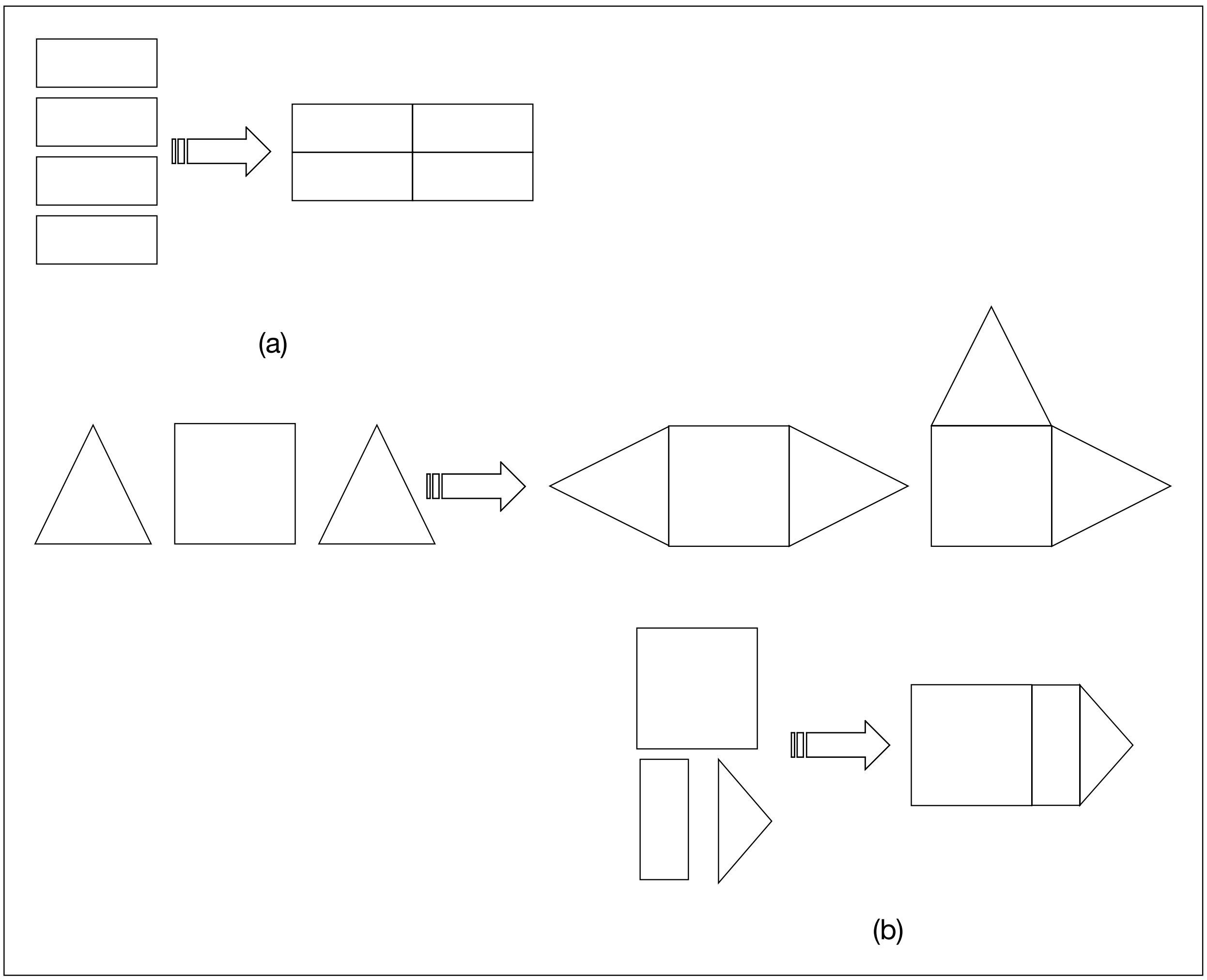


Fig. 4.2. Examples of combining shapes to make other shapes

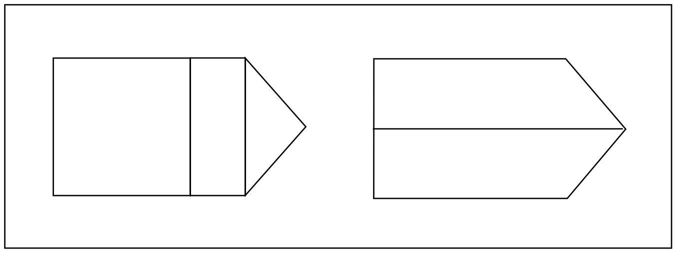


Fig. 4.3. Two different ways to combine shapes to make the same final shape

Experiences with a variety of materials—for example, pattern blocks, geoboards, and such technology as Shape Cutter (found on NCTM's Illuminations Web site)—and drawings give students opportunities to combine shapes made of many different materials. Through these experiences students begin to understand that given even a few shapes, they can combine those shapes in many different ways, as shown in figure 4.4.

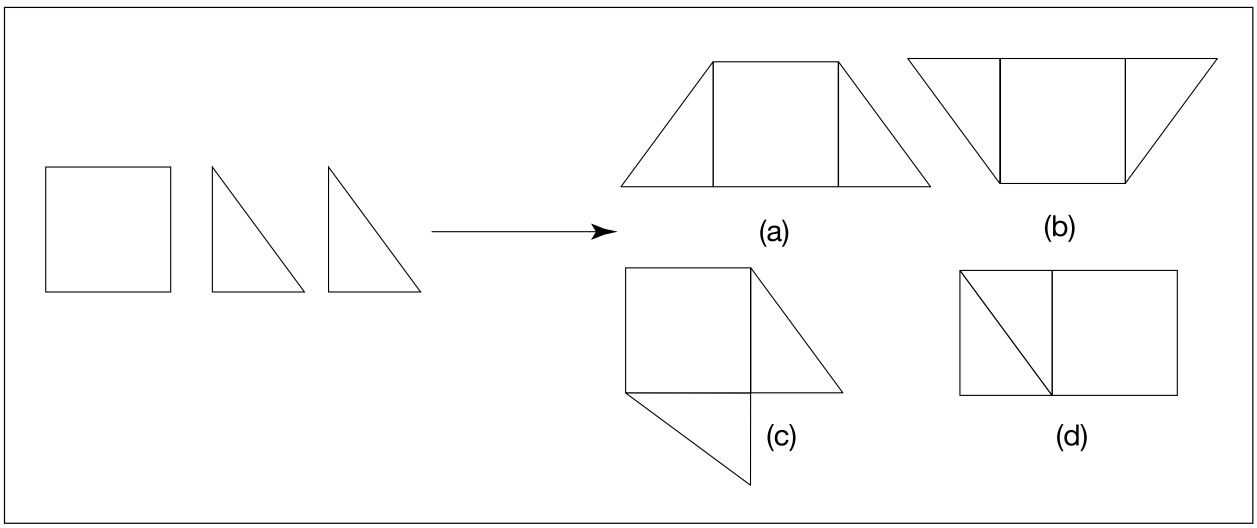


Fig. 4.4. Different ways to combine a square and two triangles

Reprinted from *Navigating through Geometry in Prekindergarten through Grade 2* (Findell et al. 2001, p. 10).

In addition to combining shapes, students learn to decompose shapes. When shapes are decomposed, they are separated into smaller shapes. At first, students may be able only to predict and visualize that when a square is divided once horizontally or vertically, the result is two rectangles, as shown in figures 4.5(a) and (b). However, as students are given more opportunities to decompose shapes, they may observe that if the square is cut once diagonally, the result is two triangles, as shown in figure 4.5(c).

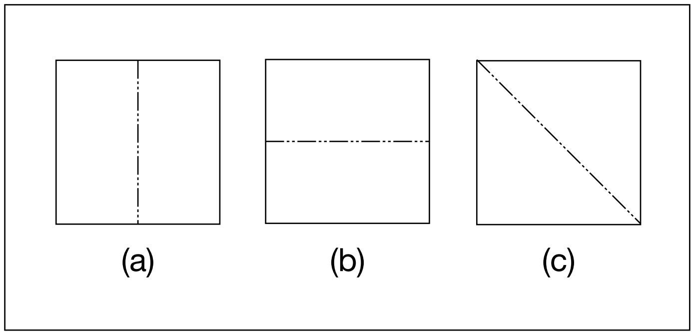


Fig. 4.5. The results of decomposing a square in different ways

Since concrete materials, such as pattern blocks, usually cannot be "broken apart" or easily decomposed, students can experience this fundamental skill of decomposition through hands-on tasks that include paper folding and the cutting apart of paper shapes or by using appropriate computer software.

As students combine and decompose shapes, they should be asked to describe and compare the characteristics of the original shape or shapes and the resulting shape or shapes to highlight which characteristics have changed and which have not. Students should consider such questions as "Does the new shape look like the original shapes? Why or why not? Has the number of sides changed? Has the number of vertices changed?" For example, in figure 4.2(a), the large rectangle looks very much like the smaller rectangles from which it was formed. However, in figure 4.2(b), the resulting five-sided shape is different from any of the shapes from which it was made.

Similar questions can be asked when students decompose shapes. For example, looking at figure 4.5, students can be led with questions to notice that in the shapes in (a) and (b), the original shape and the decomposed shapes share some of the same properties, specifically, that they are all rectangles, although the original shape is a square and the smaller, decomposed shapes are not. However, in figure 4.5(c), the properties of the original shape and those of the decomposed shapes are not the same, because the larger shape is a quadrilateral and the decomposed shapes are triangles. The visual discrimination used in analyzing the composition and decomposition of shapes will be useful to students later as they learn to find the areas of complex shapes.

In addition to extending their understanding of the properties of two-dimensional shapes, students develop and apply spatial reasoning. Although students are not introduced to formal transformation terminology until grade 4, their exposure to shapes that have been transformed by rotations (turns), reflections (flips), or translations (slides) in early geometric exploration is valuable in developing their spatial reasoning and again solidifies their understanding of the concept that orientation does not change the basic characteristics of a shape. For example, students can use paper or wooden shapes to demonstrate that the shapes in figure 4.6 are all the same, just turned in different ways.

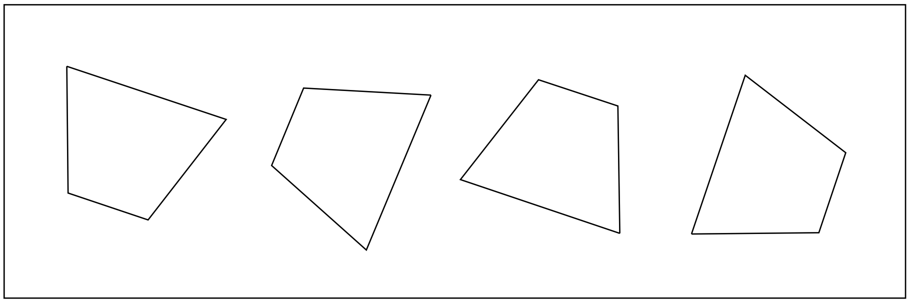


Fig. 4.6. A quadrilateral in different orientations

Through a focused curriculum in prekindergarten through grade 2, students develop the foundation for the formalization of geometric instruction in later grades. As students recognize shapes in different orientations, describe shapes by their attributes, and identify similarities and differences between shapes, they enhance their geometric reasoning and their understanding of the relationship between shapes.

Building Understanding of Two-Dimensional Shapes

Principles and Standards for School Mathematics (NCTM 2000) explains that in grade 3, students are beginning to expand their spatial and verbal reasoning powers and thus can begin to carry out more complex explorations of geometric shapes. In grade 3, students strengthen their visual discrimination skills to recognize shapes and progress from using their own language and vocabulary to using more mathematically precise vocabulary to name and describe the shapes. They gain insight into characteristics of two-dimensional shapes by decomposing, combining, and comparing shapes and use their developing understanding of two-dimensional shapes to recognize congruence and symmetry.

Using formal geometric language

Whereas in earlier grades students recognized the commonalities and differences among various shapes, in grade 3 they begin to use more formal language to describe and analyze shapes as, for example, specific polygons based on their numbers of sides and vertices. Any formal discussion of the classification of geometric shapes typically begins with a discussion of polygons. Many of the shapes that students have previously encountered are polygons, but in grade 3, they learn to use the word *polygon* to describe any closed figure with straight sides that intersect only at their endpoints. Sets of figures such as those shown in figure 4.7 can be used to highlight the distinctions between polygons and figures that are not polygons. For example, a circle is a closed shape, but students learn that it is not a polygon because it is not made up of straight sides.

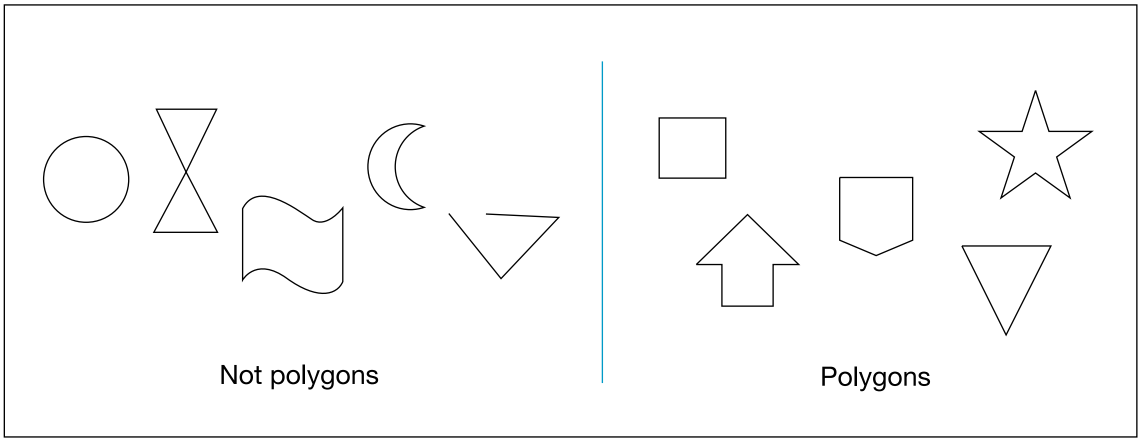


Fig. 4.7. Figures that are and are not polygons

Another example of the formalization of vocabulary in grade 3 is the move from using the word *corner* to using the word *angle,* or *vertex* if they mean the corner's "tip"—the point or intersection of two sides. As students learn to classify two-dimensional shapes, they can use the number of sides and the number of vertices as defining characteristics.

Classifying polygons

After students establish that some shapes are polygons and some are not polygons, they begin a more formal process of sorting and classifying two-dimensional shapes by examining their characteristics, such as the number of sides in, or the kinds of angles related to, a shape. Number of sides is often the most obvious characteristic that students will notice about a polygon. By sorting polygons according to the number of sides, students can learn the names for polygons, for example, *triangle* (three-sided polygon), *quadrilateral* (four-sided polygon), *pentagon* (five-sided polygon), and *hexagon* (six-sided polygon). Students should be asked to find examples of these types of shapes in the world around them, perhaps even collecting as many types of a shape as they can find. For example, students should recognize the shape in figure 4.8 as a hexagon, a six-sided polygon.

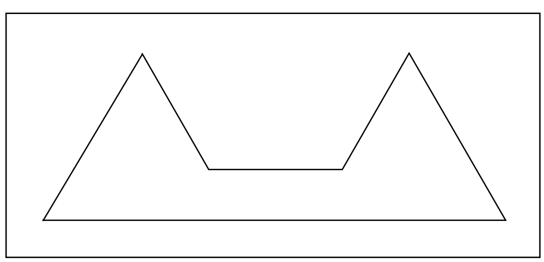


Fig. 4.8. An example of a hexagon

When students are given a variety of examples such as those illustrated in figure 4.9, they can see that many different "corners" or "angles" appear in two-dimensional shapes.

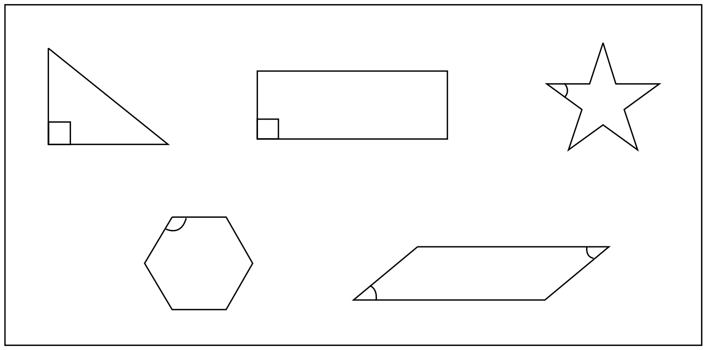


Fig. 4.9. A variety of angles in two-dimensional shapes

Through their experiences with shapes, students learn that the kind of angle found at the corner of a rectangle is a special kind of angle, a *right* angle. They will eventually use the shape of the right angle as a benchmark to describe other angles: ones that are less open than a right angle, called *acute* angles, and ones that are more open than a right angle, called *obtuse* angles. Students' understanding of these characteristics of angles becomes essential to their ability to identify, classify, and describe geometric shapes. For example, with appropriate questioning, students realize that rectangles, including squares, are distinguishable from other quadrilaterals because of their four right angles.

As students move into a more in-depth study of the relationships among geometric shapes, they move beyond recognizing the identity of a shape on the basis of the number of sides and angles that it has to making more general inferences. One such generalization is that all polygons have the same number of sides as they have angles. For example, all three-sided polygons have three angles, all four-sided polygons have four angles, all five-sided polygons have five angles, and so on, as illustrated in figure 4.10. Thus, students can classify a polygon on the basis of their knowledge of either the number of sides or the number of angles. For example, if a student knows that a polygon has three angles, he or she can infer that it is a triangle and that it has three sides. If a student knows that a polygon has six sides, he or she can infer that it has six angles and is a hexagon.

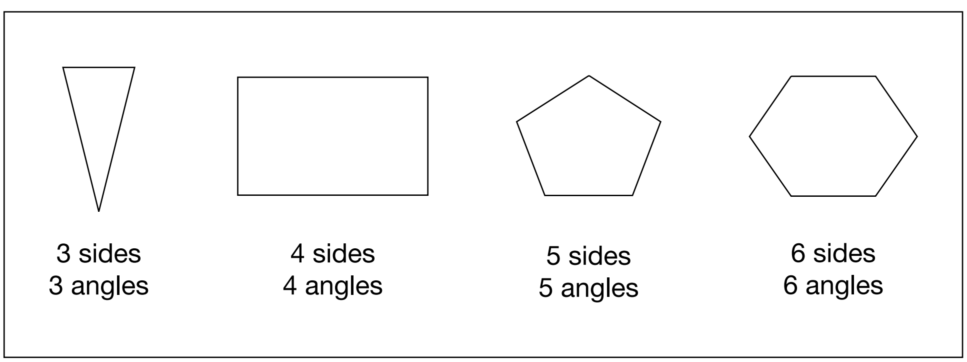


Fig. 4.10. Identifying the number of sides and angles in different polygons

Having students model and build two-dimensional shapes with such materials as geoboards, craft sticks, and straws will help students consolidate their understanding of geometric properties. Such construction activities give students opportunities to test their theories about the characteristics, or attributes, of shapes and help integrate students' previous understandings of the new grade 3 concepts they are learning. Concrete exploration of geometry will also help students continue to develop their understanding that orientation, size, and material have no effect on the properties of shapes. Once students have built shapes out of a variety of materials, they can apply their understanding to drawing many different examples of circles, squares, triangles, rectangles, trapezoids, and so on.

Classifying triangles and quadrilaterals

Once students have a firm understanding of how to classify polygons by the numbers of sides and angles, they can move on to a more sophisticated classification method. Polygons can be classified by the types of angles they contain or by the relationships among their sides. For example, through previous explorations with concrete materials, students have already developed an understanding that different types of triangles exist. As students' study of triangles continues, they learn that they can classify those different triangles in different ways.

One way students can classify triangles is by the types of interior angles in the triangle. *Right triangles* have one right angle. Thus, students can refine their classification of a polygon that has exactly three angles, one of which is a right angle, not only as a triangle but as a right triangle.

Students can also classify triangles by the relative lengths of their sides. A triangle with all three sides having equal length is *equilateral;* a triangle with at least two sides of the same length is *isosceles;* and triangle in which no sides are the same length is *scalene*. Thus, when presented with a three-sided polygon in which no sides have the same length, students can label it as not only a triangle but a scalene triangle.

Students at this grade level can also deepen their ability to describe quadrilaterals on the basis of the angles and relative lengths of the sides of the shapes, as well as the relationship between opposite sides. When comparing opposite sides, students develop the intuitive idea of parallel lines. Students begin to realize that the opposite sides in a rectangle belong to lines that will never meet, a geometric property of parallel lines. As students use manipulatives, drawings, and technology to explore different arrangements of parallel sides, they can observe that some quadrilaterals other than rectangles can have opposite sides that are parallel. In later grades they will learn that this entire group of quadrilaterals with two pairs of opposite parallel sides, including rectangles, is called *parallelograms*. Quadrilaterals that have only one pair of opposite parallel sides are called *trapezoids*.

Figure 4.11 shows types of quadrilaterals and some possible descriptions of each type. Through thorough examination and discussions of these and other descriptions, students become aware of the properties of each type of quadrilateral and how they are alike and different.

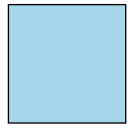
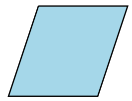
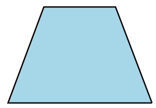
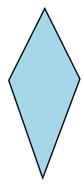
Name of Quadrilateral	Visual	Description
Rectangle		Opposite sides the same length, with four right angles
Square		All sides the same length, with four right angles
Parallelogram		Opposite sides the same length and parallel
Rhombus		All sides the same length, and opposite sides parallel
Trapezoid		Exactly one pair of parallel sides
Kite		Two pairs of adjacent sides of equal length and exactly one pair of opposite angles that are congruent

Fig. 4.11. Classifications of quadrilaterals

Once students have a grasp of the different types of quadrilaterals and their descriptions, they can begin to explore the embedded relationships among quadrilateral classifications. The Venn diagram in figure 4.12 illustrates these relationships and is a helpful tool in guiding instruction and questioning to help students clarify and deepen their understanding of quadrilaterals. For example, students could be led to generalize that all rectangles are parallelograms because they are four-sided polygons with two pairs of parallel sides. A discussion could be built around why a square is both a type of rectangle and a type of rhombus. These types of discussions allow students to reexamine their descriptions of the various quadrilaterals and the relationships among those descriptions.

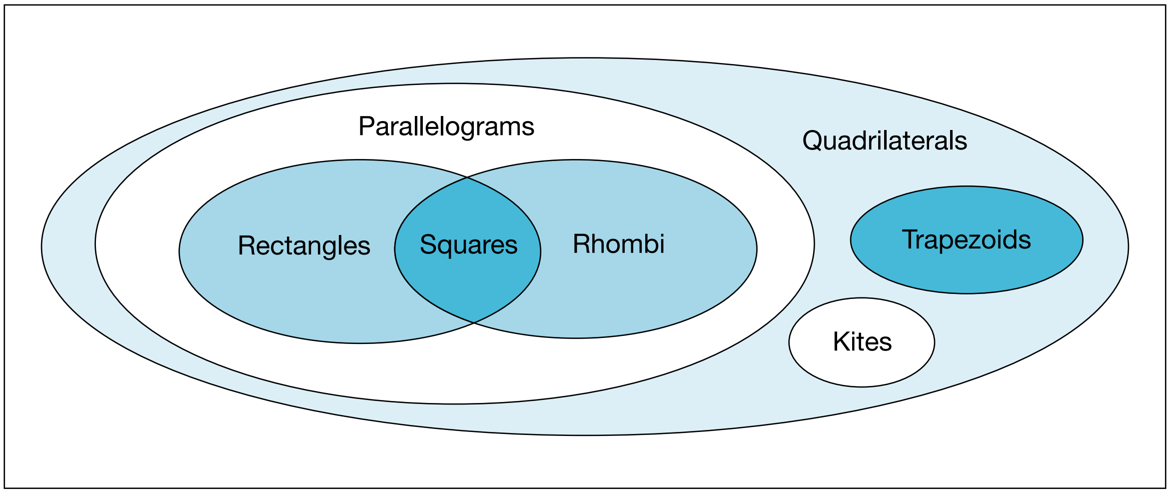


Fig. 4.12. Venn diagram illustrating the embedded relationship among quadrilaterals

In grade 3, students are just beginning to comprehend these classifications and their relationships, and they will internalize them at their own pace. Students should be given many opportunities to explore the properties of quadrilaterals, perhaps with technology, and be encouraged to think about and discuss the relationships among quadrilaterals while using details and proper vocabulary. Students will also benefit from being exposed to, and creating on their own, a variety of examples of each type of quadrilateral. The opportunities to apply and discuss these classifications will help students improve their understandings of the interconnected, embedded relationship among quadrilaterals.

Applications of combining and decomposing polygons

Students build on the ability to combine and decompose shapes learned in grade 2 as they apply these skills to other areas of mathematics learned in grade 3. One application of these skills is in the topic of fractions. For example, as less experienced students interpret the shapes in figure 4.13, they observe that the squares have been combined to create a rectangle. However, as students mature in their understanding of combining shapes, they learn that these squares have been combined in a particular way, that is, to show a rectangle divided into five equal parts. Thus, students can apply the combining of geometric shapes to fractions as they interpret the rectangle as a fraction model that shows fifths and in which each square is one-fifth of the resulting shape.

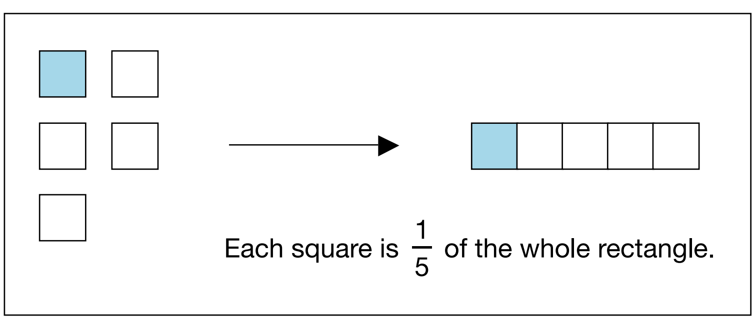


Fig. 4.13. Example illustrating how combining geometric figures can be applied to fractions

At the same time as they observe results of composing shapes to model fractions, students can learn how to apply the decomposition of geometric shapes for the same purpose. Previously, students would interpret the example in figure 4.14 as decomposing a circle into smaller parts. However, as students mature in their understanding of decomposition, they reinterpret the model as decomposing the circle into four equal parts and that each part represents 1 fourth of the circle. Thus, the decomposed circle can be interpreted as a fraction model for fourths.

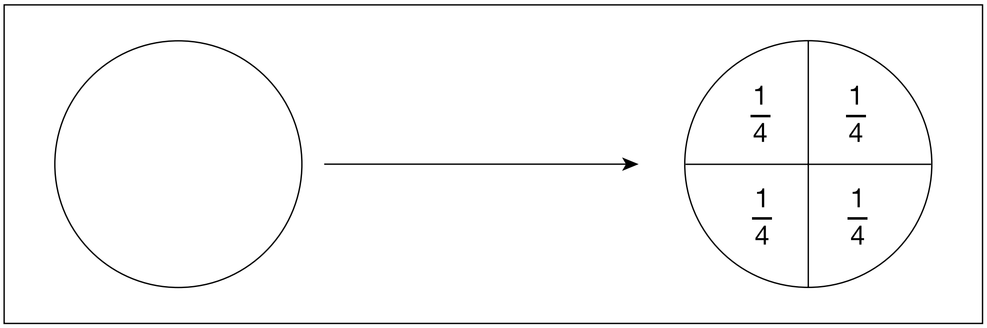


Fig. 4.14. Example illustrating how decomposing geometric figures can be applied to fractions

In grade 3, students also learn to apply combining and decomposing geometric shapes to multiplication, specifically, to help them understand the area model. In initial explorations of multiplication, students use the array, an arrangement of separate objects in equal rows and columns, as a model to show multiplication facts. For example, in figure 4.15, the model is a two-by-four array consisting of eight squares in all. At first, students count the number of squares to find the number in all. As students link their understanding of combining and decomposing shapes with models for multiplication, they begin to see that the model is, in fact, similar to a rectangle decomposed into rows of adjacent squares. They also learn that they can multiply the number of rows by the number of columns to find the total number of squares in all. Thus, the area model can be viewed either as a combination of squares from an array or as a rectangle decomposed into an array of squares, illustrating that two groups of four, or 2 × 4, equals eight. Students will revisit this model when they learn about area as a measurable attribute.

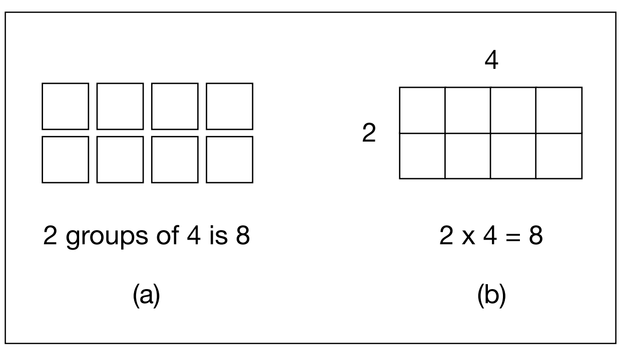


Fig. 4.15. Illustration of how combining and decomposing shapes are related to the area model for multiplication

Students can also apply their understanding of arrays as decomposed rectangles to multiplication properties. For example, students can decompose and combine shapes to show the commutative property of multiplication. As illustrated in figure 4.16, students can see that the rectangle can be decomposed to show an array having four rows with three squares in each row. Then these squares can be combined in another way to show an array of the same size and shape and that has three rows with four squares in each row. Thus, if students know that they can multiply the number of rows by the number of squares in each row to get the total number of squares in an array, they can use the decomposition and combining in this model to show that 4 × 3 = 3 × 4.

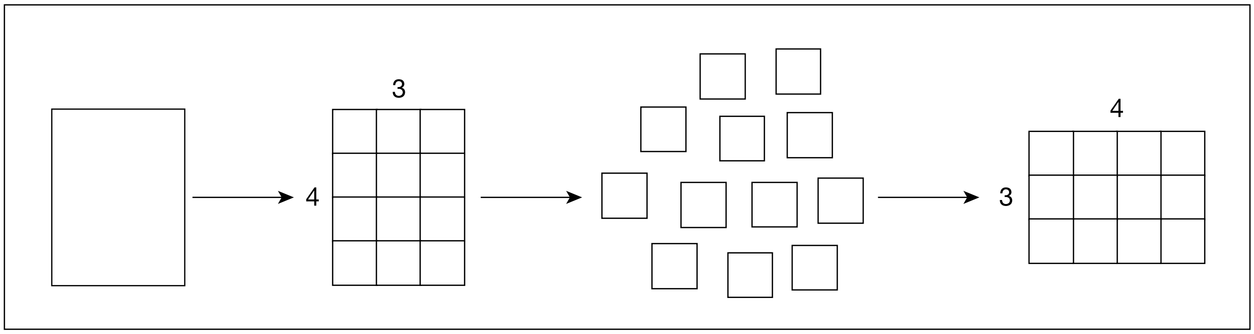


Fig. 4.16. Example illustrating how decomposition and combining can be applied to the commutative property of multiplication

Students can also use decomposition and combining to illustrate the distributive property of multiplication over addition. As illustrated in figure 4.17, students can see that because the rectangle has been decomposed into an array with four rows and three squares in each row, it can be decomposed further to show two rectangles, each with two rows and three squares in each row. Thus, the model shows that 4 × 3 = (2 + 2) × 3 = (2 × 3) + (2 × 3).

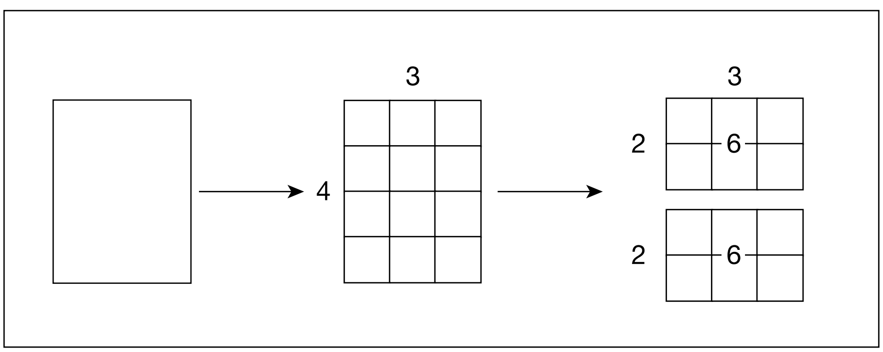


Fig. 4.17. Example illustrating how decomposition can be applied to the distributive property of multiplication over addition

As students continue to develop their ability to apply combining and decomposing shapes to multiplication models, they simultaneously begin to build the foundation for understanding the area concepts that are introduced in grade 4. Figure 4.15 illustrates how students will begin to apply this same model in their exploration of area. Their understanding that some rectangles can be easily decomposed into squares of the same size becomes the foundation for one of the most crucial concepts in area, specifically, the idea that each of those squares is a square unit and that the area of a shape is the number of square units needed to cover the shape. As students learn to internalize the concept of area, they will begin to interpret the model in figure 4.15 as showing a rectangle with an area of 8 square units.

Exploring congruence

The concept of "same size, same shape" is a relatively intuitive notion that students begin to develop very early, outside of any formal study of geometry. The concept is formalized in grade 3 when students learn that when two shapes are the same size and shape, they are congruent. The concept of congruence is interwoven throughout almost every geometric concept that students encounter. For example, to classify polygons, students consider whether the sides of the shape are congruent (the same length) and whether the angles are congruent (open the same amount). To identify the properties of a shape, students also consider congruence. For example, a shape is a rhombus if all its sides are the same length, or congruent, and opposite angles are the same size, or congruent; and a triangle is isosceles if at least two of its sides are congruent. Later, as students begin to explore transformations, they learn that when shapes are transformed through translations (slides), rotations (turns), or reflections (flips), the resulting shape is congruent to the original shape, although its orientation might be changed.

Students in grade 3 can investigate the idea of congruence by inspection of shapes. However, allowing students to cut out shapes and place them on top of each other to determine congruence will help students solidify their understanding that congruence is not based on orientation and that congruent shapes have exactly the same geometric properties, for example, the same number of sides and angles that are the same size. Students need to realize that neither inspection nor cutting out and moving shapes actually proves that two shapes are congruent, since our senses are not that accurate. However, these techniques can help students understand the meaning of congruence. Understanding how to apply the vocabulary of congruence to the concept of "same size, same shape" enables students to mature in their ability to describe properties of two-dimensional shapes.

Exploring line symmetry

Students apply their developing understanding of congruence as they begin their study of a new, yet intuitive geometric concept—*symmetry*. In early discussions, students learn that if a line can be drawn through a shape that divides the shape into two congruent parts that are oriented in such a way as to make mirror images of each other, as shown in figure 4.18, then the shape has line symmetry, and the line is called a *line of symmetry*. The shapes at the left in the figure have at least one line of symmetry, so they have line symmetry. The shapes at the right do not have such a line, so they do *not* have line symmetry.

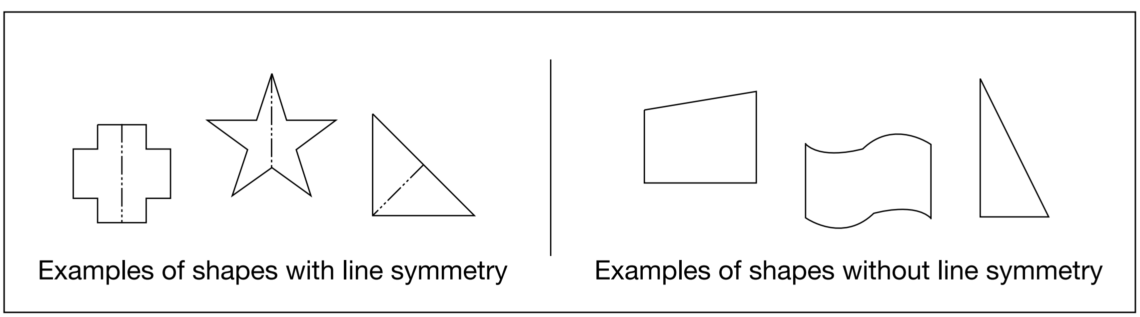


Fig. 4.18. Examples of shapes with and without line symmetry

Students can hypothesize by inspection whether a shape has a line of symmetry. However, students who are given opportunities to use paper folding to identify possible lines of symmetry will develop a deeper understanding of the concepts of symmetry. Through these experiences, students observe that if a folded shape has a line of symmetry, the edges of the folded parts will match exactly, as shown in figure 4.19.

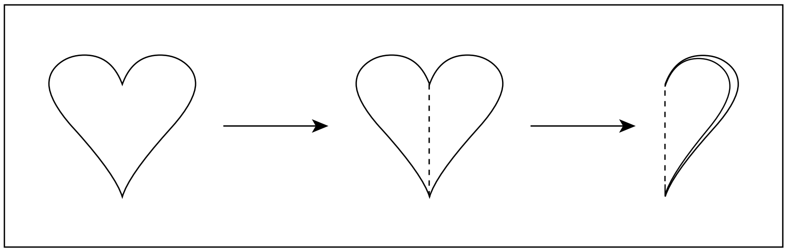


Fig. 4.19. Illustration of how folding shapes can test for symmetry

Students can also use mirrors to understand and identify symmetry. For example, students can place a mirror over the hypothesized line of symmetry. Then they can view the reflection of the shape in the mirror. If the shape in the reflection is exactly the same as the other side of the shape, the line on which the mirror is placed is a line of symmetry, and the figure has line symmetry.

Just as they did in working with congruence, students need to realize that inspection, paper folding, and mirrors do not actually prove that a figure has line symmetry, because our senses are not exact. However, these techniques can help students understand the meaning of line symmetry.

Whether students use inspection, paper folding, or mirrors to identify symmetry, they will begin to notice that shapes can have zero lines of symmetry, one line of symmetry, two lines of symmetry, and so forth, to infinitely many lines of symmetry, as illustrated in figure 4.20.

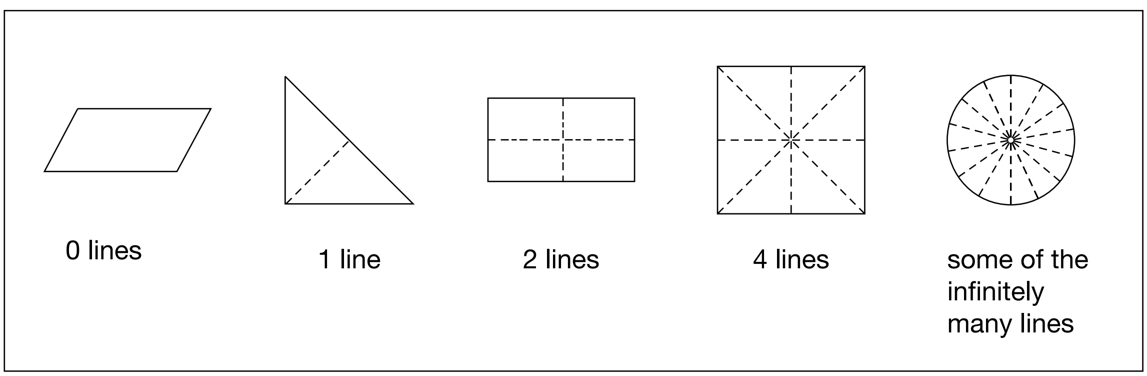


Fig. 4.20. Examples of lines of symmetry

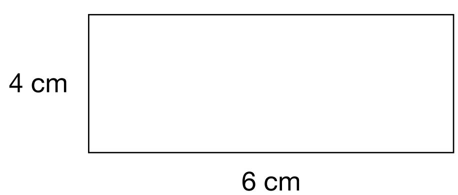
In later grades, as students connect line symmetry with reflections, they will observe that they can identify a shape with symmetry if, when reflected over a line (the line of symmetry), the resulting image coincides with ("lands" in the same position as) the original shape. Through experiences with concrete and pictorial materials, students will also be able to connect their understanding of symmetry with the idea of congruence.

Strengthening Fluency through Connections

Understanding connections among mathematics concepts leads students to a deeper understanding of mathematics and the interrelationship among concepts presented both within a grade level and across grade levels. Connections with other mathematics concepts at this grade level are evident throughout this Focal Point.

As discussed previously, students connect the geometric concepts learned in grade 3 with multiplication and fractions as they use combining and decomposing shapes to model fractions, reconceptualize arrays, and illustrate multiplication properties. Thus, geometry provides the vehicle through which students can make sense of concrete and pictorial models as representations of abstract concepts, allowing students to link the new concepts with their prior knowledge.

In addition, students can make connections between their work with geometric shapes and the work they did with linear measure in grade 2 to develop the concept of perimeter. In grade 3, students learn that perimeter is the distance around a shape, so it is a measurement of length. Through their understanding of the properties of geometric shapes and length, students can select appropriate units, strategies, and tools to solve problems involving perimeter. These solutions might first involve wrapping string around a shape and then measuring the length of the string or measuring each side with a ruler and adding the measurements. Students can then relate the grade 3 focus on analyzing, describing, comparing, and classifying properties of shapes to connect the concept of perimeter with the properties of a given shape. For example, as shown in figure 4.21, students who understand properties of rectangles know that opposite sides of a rectangle have the same length. Students can then infer from the rectangle that the sides that are not labeled are 4 cm and 6 cm. Thus, they can add 6 + 6 + 4 + 4 to find the perimeter. In later grades, we want students to be able to generalize this idea to all rectangles and express the perimeter of any rectangle as $l + l + w + w$, or $2l + 2w$.



$P = 6 + 6 + 4 + 4 = 20$ cm $\longrightarrow$ $P = l + l + w + w$
$P = 2l + 2w$

Fig. 4.21 Connecting the concept of perimeter with the properties of a rectangle

Another important connection that students make in grade 3 is between congruence and fractions. Students learn to use rectangles divided into equal parts to model fractions. After students have learned about congruence, this partitioning takes on a deeper meaning. An equal partitioning of a shape is a decomposing of it into congruent shapes, as illustrated in figure 4.22. The pentagon has been decomposed into five congruent triangles. Each triangular piece is one-fifth of the whole pentagonal region. Thus, students' understanding of congruence and decomposition can enhance their understanding of fractions.

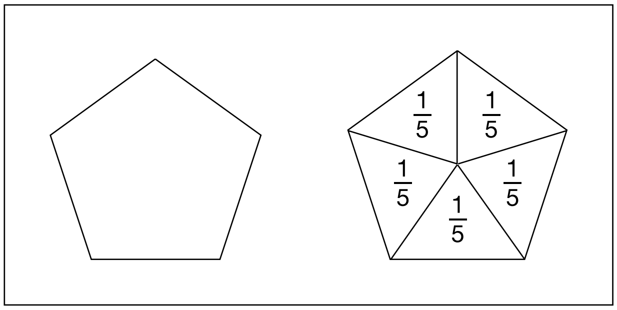


Fig. 4.22. Illustration of decomposing one shape into several congruent parts

Connections in later grades

As grade 3 students learn and analyze the properties of two-dimensional shapes, they are laying the foundation that will enable them to extend this understanding to the study of area in grade 4. For example, in grade 3, students learn how to decompose a rectangle into arrays of adjacent squares as shown in figure 4.23 and then find the number of squares in the arrays. In grade 4, students will learn that the area of a shape is the number of congruent squares needed to cover the figure. If, for example, the squares represent square inches, the area of the rectangle is 15 square inches. If the squares represent square meters, the area of the rectangle is 15 square meters. As students apply the array model and multiplication to area, they progress from counting the square units in the shape individually to multiplying the number of square units in each row (the length or width) by the number of rows of square units (the width or length) to find the area of the rectangle.

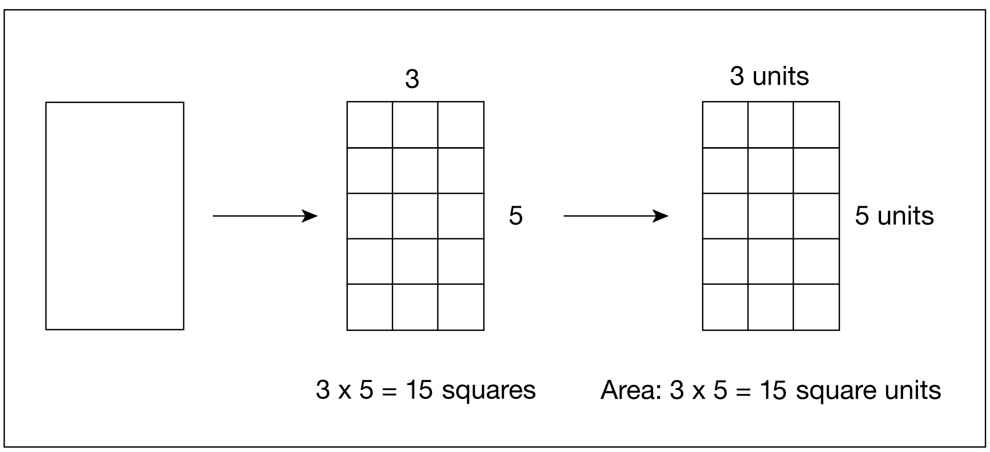


Fig. 4.23. The conceptual progression of area from decomposition to area formula

Once students have begun to understand the meaning of area in relation to rectangles, they can apply this understanding to finding the area of other polygons by decomposing and recombining the polygon into familiar shapes, such as rectangles. They can then find the areas of the parts of the original shape and add those areas to find the area of the entire polygon. This understanding lays the foundation for understanding and correctly applying the formulas for finding the areas of polygons in grades 4 and 5. For further discussion of this connection, see the Focusing on Area section in *Focus on Grade 4* (NCTM 2009).

In grade 4, students also build on their understanding of symmetry and congruence related to transformations. Students learn that if they rotate (turn), translate (slide), or reflect (flip) a shape, the resulting shape is congruent to the original shape and that line symmetry and rotational symmetry are related to reflections and rotations, respectively.

When students develop a firm understanding of the grade 3 geometry concepts, they have laid the foundation for the development of more sophisticated concepts in later grades.

Developing Depth of Understanding

Many of the concepts in the Geometry Focal Point for grade 3 rely on hands-on experiences. How can you use and integrate technology to develop and foster students' understanding of concepts presented in this Focal Point?

References

Clements, Douglas H. "Longitudinal Study of the Effects of Logo Programming on Cognitive Abilities and Achievement." *Journal of Educational Computing Research* 3 (1987): 73–94.

Donovan, M. Suzanne, and John D. Bransford, eds. *How Students Learn: Mathematics in the Classroom*. Washington, D.C.: National Research Council, 2005.

Duncan, Natalie N., Charles Geer, DeAnn Huinker, Larry Leutzinger, Ed Rathmell, and Charles Thompson. *Navigating through Number and Operations in Grades 3–5*. *Principles and Standards for School Mathematics* Navigations Series. Reston, Va.: National Council of Teachers of Mathematics, 2007.

Findell, Carol R., Marian Small, Mary Cavanagh, Linda Dacey, Carole E. Greenes, and Linda Jensen Sheffield. *Navigating through Geometry in Prekindergarten–Grade 2*. *Principles and Standards for School Mathematics* Navigations Series. Reston, Va.: National Council of Teachers of Mathematics, 2001.

Fuson, Karen C., and Aki Murata. "Integrating the NRC Principles and the NCTM Process Standards: Cognitively Guided Teaching to Individualize Instruction within Whole-Class Activities and Move All Students within Their Learning Path." *National Council of Supervisors of Mathematics Journal* 10 (Spring 2007): 72–91.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. *Adding It Up: Helping Children Learn Mathematics*. Washington, D.C.: National Research Council, 2001.

Lampert, Magdalene. "Choosing and Using Mathematical Tools in Classroom Discourse." In *Advances in Research on Teaching*, vol. 1, edited by Jere Brophy, pp. 223–64. Greenwich, Conn.: JAI Press, 1989.

Mack, Nancy K. "Learning Fractions with Understanding: Building on Informal Knowledge." *Journal for Research in Mathematics Education* 21(January 1990): 16–32.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

———. *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

———. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*. Reston, Va.: NCTM, 2006.

———. *Focus in Grade 4*. Reston, Va.: NCTM, 2009.

———. *Focus in Grade 5*. Reston, Va.: NCTM, 2009.

———. *Focus in Grade 2*. Reston, Va.: NCTM, forthcoming.

Focus in Grade 3

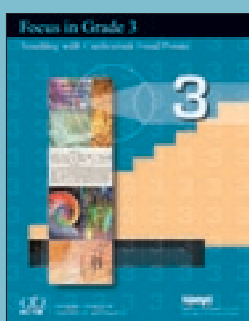
Teaching with Curriculum Focal Points

Focus in Grade 3, one in a series of grade-level publications, is designed to support teachers, supervisors, and coordinators as they begin the discussion of a more focused curriculum across and within prekindergarten through eighth grade, as presented in NCTM's Curriculum Focal Points. Additionally, teacher educators should find it useful as a vehicle for exploring mathematical ideas and issues involving the grade 3 mathematics curriculum with their preservice teachers.

This book describes and illustrates learning paths for the mathematical concepts and skills identified in each third-grade Focal Point. It includes problems for students and powerful representational supports for teaching and learning that can facilitate understanding, stimulate productive discussions about mathematical thinking, and provide a foundation for fluency with the core ideas. It also discusses common student errors and misconceptions, reasons why the errors may arise, and teaching methods or visual representations to address the errors. Because learning paths cut across grades, some discussion of related Focal Points at grades 1 and 2 is included to describe and clarify prerequisite knowledge and show how grade 3 understanding builds on what went before.

Teachers should find these discussions of the learning paths, representations, problems, and lines of reasoning valuable as they plan activities and discussions for their students and strive to help students achieve the depth of understanding of important mathematical concepts necessary for their future success.

Teaching with Curriculum Focal Points Titles in This Grade Band



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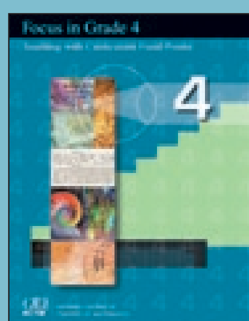

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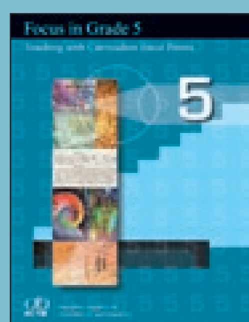

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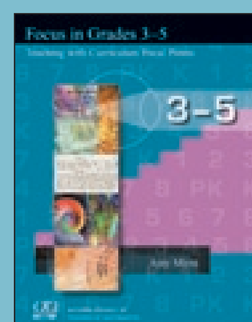

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