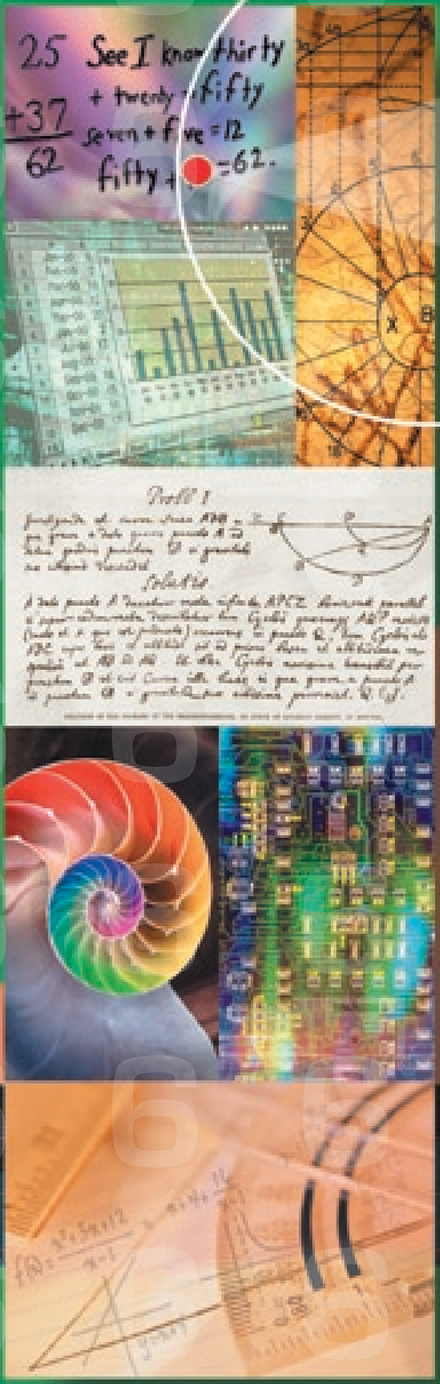
Focus in Grade 6

Teaching with Curriculum Focal Points

6



NCTM

NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in Grade 6

Teaching with Curriculum Focal Points

Additional titles in the Teaching with Curriculum Focal Points series:

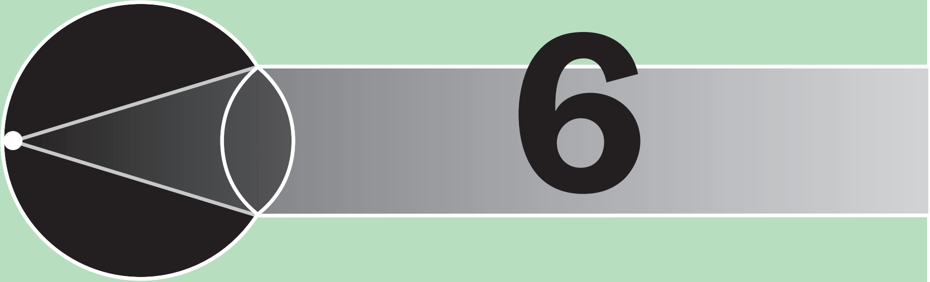
The Teaching with Curriculum Focal Points series consists of grade-level publications designed to support teachers, supervisors, and coordinators as they begin the discussion of a more focused curriculum across and within prekindergarten through grade 8, as presented in *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics*.

	ISBN #	NCTM stock #
Focus in Prekindergarten	*978-0-87353-644-8*	*13626*
Focus in Kindergarten	*978-0-87353-645-5*	*13627*
Focus in Grade 1	*978-0-87353-646-2*	*13628*
Focus in Grade 2	*Coming Fall 2010*	
Focus in Pre-K–2	*978-0-87353-624-0*	*13486*
Focus in Grade 3	*978-0-87353-625-7*	*13487*
Focus in Grade 4	*978-0-87353-627-1*	*13490*
Focus in Grade 5	*978-0-87353-614-1*	*13437*
Focus in Grades 3–5	*978-0-87353-609-7*	*13395*
Focus in Grade 7	*978-0-87353-649-3*	*13631*
Focus in Grade 8	*978-0-87353-650-9*	*13632*
Focus in Grades 6–8	*978-0-87353-618-9*	*13465*

Please visit www.nctm.org/catalog for details and ordering information.

Focus in Grade 6

Teaching with Curriculum Focal Points



Jane F. Schielack, *Series Advisor*
Texas A&M University

NCTM
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, INC.
1906 Association Drive, Reston, VA 20191-1502
(703) 620-9840; (800) 235-7566; www.nctm.org

Library of Congress Cataloging-in-Publication Data
Focus in grade 6 : teaching with curriculum focal points.
p. cm.
Includes bibliographical references.
ISBN 978-0-87353-648-6
1. Mathematics—Study and teaching (Elementary)—United States—Standards. 2. Sixth grade (Education) —Curricula—United States—Standards. 3. Curriculum planning—United States—Standards.
QA135.6.F6294 2010
372.7—dc22
2009040432

The National Council of Teachers of Mathematics is a public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

e-Book ISBN: 978-0-87353-794-0
Printed in the United States of America

Contents

Preface vii

Preface to *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* viii

Acknowledgments ix

1. Introduction 1

- Purpose of This Guide 1
- Purpose of Curriculum Focal Points 1
- The Impact of Focal Points on Curriculum, Instruction, and Assessment 2
- Using This *Focus in Grade 6* Book 2
- Bringing Focus into the Classroom: Instruction That Builds Understanding and Fluency 3
  - An instructional progression approach 3
  - In-depth instructional conversations 4
  - Using mathematical representations 4
- An Important Grade 6 Issue: Developing Fluency with Multiplication and Division of Fractions and Decimals 5
- Curriculum Focal Points and Connections for Grade 6 6

2. Focusing on Multiplication and Division of Fractions and Decimals 7

- Instructional Progression for Multiplication and Division of Fractions and Decimals 7
- Early Foundations in Multiplication and Division of Fractions and Decimals 7
- Early Foundations in Whole-Number Multiplication and Division 8
- Early Foundations in Understanding Fractions and Decimals 9
- Building Understanding of, and Fluency with, Multiplication and Division of Fractions and Decimals 10
- Fractions 11
  - Multiplying fractions 11
  - Modeling multiplication with fractions 12
  - Procedures for multiplying fractions 18
  - Dividing fractions 20
  - Modeling division with fractions 22
  - Modeling measurement division 22
  - Modeling partitive division 25
  - Procedures for dividing fractions 30
- Decimals 31
  - Fraction and decimal equivalence 31
  - Place-value patterns in multiplying and dividing by powers of 10 33
  - Multiplication with decimals 34

Contents — Continued

Division with decimals ... 36
Strengthening Multiplication and Division of Fractions and Decimals through Problem Solving ... 40
Strengthening Understanding through Connections ... 42
Connections in later grades ... 44
Developing Depth of Understanding ... 44

3. Focusing on Ratio and Rate ... 45
Instructional Progression for Ratio and Rate ... 45
Early Foundations in Ratio and Rate ... 45
Focusing on Ratio and Rate ... 47
Moving from additive to multiplicative reasoning ... 47
Using fractions to build an understanding of ratios ... 49
Using equivalent fractions to build an understanding of equivalent ratios ... 57
Understanding rate as a special kind of ratio ... 62
Strengthening Connections among Multiplication, Division, Fractions, and Ratios through Problem Solving ... 66
Strengthening Understanding of Ratio, Rate, and Unit Rate through Connections ... 69
Connections in later grades ... 72
Developing Depth of Understanding ... 73

4. Focusing on Expressions and Equations ... 75
Instructional Progression for Expressions and Equations ... 75
Early Foundations in Mathematical Expressions and Equations ... 75
Focusing on Writing, Interpreting, and Using Expressions and Equations ... 77
Using language to develop understanding of expressions and equations ... 79
Developing expressions and equations to represent a sequence ... 82
Strengthening Understanding of Expressions and Equations through Problem Solving ... 93
Strengthening Understanding of Expressions and Equations through Connections ... 97
Connections in later grades ... 98
Developing Depth of Understanding ... 99

References ... 100

PREFACE

On September 12, 2006, the National Council of Teachers of Mathematics released *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* to encourage discussions at the national, state, and district levels on the importance of designing a coherent elementary mathematics curriculum focusing on the important mathematical ideas at each grade level. The natural question that followed the release of *Curriculum Focal Points* was "How do we translate this view of a focused curriculum into the classroom?"

Focus in Grade 6, one in a series of grade-level publications, is designed to support teachers, supervisors, and coordinators as they begin the discussion of a more focused curriculum across and within prekindergarten through grade 8, as presented in *Curriculum Focal Points*. *Focus in Grade 6*, in conjunction with the *Focus in Grade 7* and *Focus in Grade 8* books, will provide a strong foundation for mathematics in a focused curriculum across grades 6 through 8. Important mathematics to prepare students for grade 6 is addressed in the NCTM publications *Focus in Grade 3*, *Focus in Grade 4*, and *Focus in Grade 5*. Additionally, teacher educators should find *Focus in Grade 6* useful as a vehicle for exploring with their preservice teachers the mathematical ideas and curriculum issues related to the suggested grade 6 Curriculum Focal Points.

The contributors to, and reviewers of, these publications, all active leaders in mathematics education and professional development, guided the creation of this grade-level book as a framework for lesson-study experiences in which teachers deepen their understanding of the mathematical ideas they will be teaching. This book describes and illustrates instructional progressions for the mathematical concepts and skills of each grade 6 Curriculum Focal Point, including powerful representational supports for teaching and learning that can facilitate understanding, stimulate productive discussions about mathematical thinking, and provide a foundation for fluency with the core ideas. Because these instructional progressions cut across grades, you will see the progressions in each grade accompanied by summaries of progressions before and after that grade that connect to Focal Points and connections in previous and following grades.

Whether you are working with your colleagues or individually, we hope you will find the discussions of the instructional progressions, representations, problems, and lines of reasoning valuable as you plan activities and discussions for your students and as you strive to help your students achieve the depth of understanding of important mathematical concepts necessary for their future success.

—*Jane F. Schielack*
Series Advisor

PREFACE TO *CURRICULUM FOCAL POINTS FOR PREKINDERGARTEN THROUGH GRADE 8 MATHEMATICS*

To address the need for a prototypical, coherent, grade-level-specific mathematics curriculum linked to *Principles and Standards for School Mathematics* (NCTM 2000), the National Council of Teachers of Mathematics asked a team of mathematicians, mathematics educators, and school-based educators to identify three or four focal points in mathematics for each grade level, prekindergarten through grade 8. The writing team—consisting of at least one university-level mathematics educator or mathematician and one pre-K–8 classroom practitioner from each of the three grade bands (pre-K–grade 2, grades 3–5, and grades 6–8)—worked together to create a set of focal points that could serve as areas of emphasis for each grade level and be used as an outline for an articulated pre-K–8 mathematics curriculum. The members of the writing team based their decisions on recommendations from *Principles and Standards*, examinations of multiple curricula from several states and countries, and reviews of a wide array of researchers' and experts' writings on the subject.

We appreciate the contributions of all who have made this document possible.

On behalf of the Board of Directors,

Cathy Seeley
President, 2004–2006
National Council of Teachers of Mathematics

Francis (Skip) Fennell
President, 2006–2008
National Council of Teachers of Mathematics

Members of the Curriculum Focal Points for Grades PK–8 Writing Team

Jane F. Schielack, *Chair,* Texas A&M University, College Station, Texas
Sybilla Beckmann, University of Georgia, Athens, Georgia
Randall I. Charles, San José State University (emeritus), San José, California
Douglas H. Clements, University at Buffalo, State University of New York, Buffalo, New York
Paula B. Duckett, District of Columbia Public Schools (retired), Washington, D.C.
Francis (Skip) Fennell, McDaniel College, Westminster, Maryland
Sharon L. Lewandowski, Bryant Woods Elementary School, Columbia, Maryland
Cathy Seeley, Charles A. Dana Center, University of Texas at Austin, Austin, Texas
Emma Treviño, Charles A. Dana Center, University of Texas at Austin, Austin, Texas
Rose Mary Zbiek, The Pennsylvania State University, University Park, Pennsylvania

Staff Liaison

Melanie S. Ott, National Council of Teachers of Mathematics, Reston, Virginia

ACKNOWLEDGMENTS

The National Council of Teachers of Mathematics would like to thank the following individuals for developing a detailed outline of the content of this publication and for their reviews of, and feedback on, drafts of the manuscript. Special thanks go to Janie Schielack for all her time and support, her invaluable guidance and advice, and her continuing commitment to the Curriculum Focal Points project.

Series Advisor
Jane F. Schielack

Content Development
Words & Numbers
Baltimore, MD

Developers
Gladis Kersaint
University of South Florida

Connie Laughlin
Milwaukee, WI

Jim Lewis
University of Nebraska—Lincoln

Reviewers
Rose Mary Zbiek
Pennsylvania State University

Marshalyn Baker
Messalonskee Middle School
Oakland, Maine

David Masunaga
Iolani School
Honolulu, HI

1 Introduction

Purpose of This Guide

A curriculum is more than a collection of activities: It must be coherent, focused on important mathematics, and well articulated across the grades.

—The Curriculum Principle, *Principles and Standards for School Mathematics*

Your first question when looking at NCTM's Curriculum Focal Points might be "How can I use NCTM's Focal Points with the local and state curriculum I am expected to teach?" NCTM's Curriculum Focal Points are not intended to be a national curriculum but have been developed to help bring more consistency to mathematics curricula across the country. Collectively, they constitute a framework of how curricula might be organized at each grade level, prekindergarten through grade 8. They are also intended to help bring about discussion within and across states and school districts about the important mathematical ideas to be taught at each grade level. Because of the current variation among states' curricula, the Curriculum Focal Points are not likely to match up perfectly with any one state's curriculum. This volume explores the mathematics emphasized at grade 6 in the focused curriculum suggested by the NCTM Curriculum Focal Points framework. Additional grade-level and grade-band books are available from NCTM to help teachers translate the Curriculum Focal Points identified for their grade level into coherent and meaningful instruction. Taken together, this grade 6 guide, along with the grades 5, 7, and 8 guides and the grades 3–5 (Mirra 2008) and 6–8 (Mirra 2009) grade-band guides, can be used by groups of teachers in professional development experiences as well as by individual classroom teachers.

Purpose of Curriculum Focal Points

The mathematics curriculum in the United States has often been characterized as a "mile wide and an inch deep." Many topics are studied each year—often reviewing much that was covered in previous years—and little depth is added each time the topic is addressed. In addition, because education has always been locally controlled in the United States, learning expectations can significantly differ by state and local school systems. NCTM's *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (2006) is the next step in helping states and local districts refocus their curricula. It provides an example of a focused and coherent curriculum in prekindergarten through grade 8 by identifying the most important mathematical topics, or "Focal Points," at each grade level. The Focal Points are not discrete topics to be taught and checked off, but rather clusters of related knowledge, skills, and concepts. By organizing and prioritizing curriculum and instruction in prekindergarten–grade 8 around Focal Points at each grade level, teachers can foster more cumulative learning of mathematics by students, and students' work in the later grades will build on and deepen what they learned in the earlier grades. Organizing mathematics content in this way will help ensure a solid mathematical foundation for high school mathematics and beyond.

Prior to the Curriculum Focal Points, the National Council of Teachers of Mathematics began the process of bringing about change to school mathematics programs in the 1980s, particularly with the first publication to outline standards in mathematics, titled *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). That publication provided major direction to states and school districts in developing their curricula. NCTM's *Principles and Standards for School Mathematics* (2000) further elaborated on the ideas of the 1989 Standards, outlining learning expectations in the grade bands of prekindergarten–2, 3–5, 6–8, and 9–12. *Principles and Standards* also highlighted six principles, which included the Curriculum Principle, to offer guidance for developing mathematics programs. The Curriculum Principle emphasized the need to link with, and build on, mathematical ideas as students progress through the grades, deepening their mathematical knowledge over time.

The Impact of Focal Points on Curriculum, Instruction, and Assessment

Significant improvement can be made in the areas of curriculum, instruction, and assessment by identifying Focal Points at each grade level. At the curriculum level, Focal Points will allow for more rigorous and in-depth study of important mathematics at each grade level. This rigor will translate to a more meaningful curriculum that students can understand and apply. At the instructional level, Focal Points will allow teachers to more fully know the core topics they are responsible for teaching. Teachers will not necessarily be teaching *less* or *more* but will be able to teach *better*. Professional development can also be tailored to deepen teachers' knowledge of these Focal Points and connect these ideas in meaningful ways. Assessments can be designed that truly measure students' mastery of core topics rather than survey a broad range of disparate topics, thus allowing for closer monitoring of students' development. At the level of classroom assessment, having a smaller number of essential topics will help teachers have time to better determine what their students have learned and whether they have learned the material deeply enough to use and build on it in subsequent years. If state assessments are more focused as well, more detailed information can be gathered for districts and schools on areas for improvement.

Using This *Focus in Grade* 6 Book

Many teachers tell us that they did not have an opportunity in their teacher preparation programs to build sufficient understanding of some of the mathematics topics that they now teach. The discussion of the mathematical ideas presented here is detailed enough for teachers to begin building understanding of the mathematics contained in each grade 6 Focal Point. To further understand what mathematics students are expected to learn before grade 6 and in later grades, teachers would benefit from examining the publications *Focus in Grade 5* and *Focus in Grade 7*. We suggest that teachers form study groups (such as those in lesson study or mathematics circles, or other learning communities) to read and discuss parts of this publication, to work together to build a deeper understanding of the mathematics topics in each Focal Point, and to plan how to help their students develop such understanding by adapting as needed their present grade 6 teaching and learning strategies and materials. A helpful approach for other teacher working groups has been to share students' insights and questions and to look at students' work to understand different ways students are solving problems, to address errors and misconceptions, and to help students move forward in a progression that fosters both understanding and fluency. Because teachers' lives are busy and demanding, the reader is better served by concentrating on small portions of this publication at a time and working through them deeply instead of trying to do too much at once and getting discouraged. Teachers' learning, like students' learning, is a continual process that can be very rewarding.

Bringing Focus into the Classroom: Instruction That Builds Understanding and Fluency

Although the main goal of this publication is to present in more detail the mathematical content in each of the Focal Points, some important pedagogical issues also need to be taken into account when creating an environment that supports focused instruction. Pedagogical principles for classrooms that do help students build understanding are outlined in *Principles and Standards for School Mathematics* (NCTM 2000) and in the National Research Council reports *Adding It Up* (Kilpatrick, Swafford, and Findell 2001) and *How Students Learn: Mathematics in the Classroom* (Donovan and Bransford 2005). An instructional environment that supports the development of understanding and fluency should be based on a logical progression of content that is connected across grades as well as within grades, should provide opportunities for students and teachers to engage in mathematically substantive discussions, and should involve teachers and students in interpreting and creating mathematical representations to enhance their understanding.

An instructional-progression approach

An instructional progression of concepts and skills supports coherence across and within grades. The table at the beginning of each Focal Point outlines the instructional progression and presents the mathematics suggested for grade 6 within the context of the related mathematics suggested for the grades before and after. Teacher study groups can work to identify gaps in the knowledge of their students that might be causing them difficulties with the mathematics in grade 6. In addition, the instructional progression offers a view of the future mathematics in which students will be applying the knowledge and skills learned in grade 6.

In-depth instructional conversations

Students have little opportunity to build understanding in a classroom in which the teacher does all the talking and explaining. A valuable instructional approach is one in which teachers create a nurturing, meaning-making community as students use “math talk” to discuss their mathematical thinking and help one another clarify their own mathematical thinking, understand and overcome errors, and describe the methods they use to solve problems (Fuson and Murata 2007). Such discussions identify commonalities and differences as well as advantages and disadvantages across methods. By having students talk about their own strategies, teachers can help them become aware of, and build on, their implicit informal knowledge (Lampert 1989; Mack 1990). As the teacher and students learn to listen respectfully to the math talk of others, they model, structure and clarify, instruct or explain, question, and give feedback to enhance one another’s learning. As students’ understanding and fluency in various topics increase, the amount and type of class discussion related to each topic will change. In-depth discussion of new topics should begin as more sophisticated, mature discussion of previously encountered topics continues.

Using mathematical representations

The use of mathematical representations, in particular mathematical drawings, during problem-solving discussions and explanations of mathematical thinking helps listeners better understand the speaker. The use of mathematical drawings as a component of homework and classwork by both students and the teacher helps them better understand each other’s thinking and thus provides continual assessment to guide instruction as the teacher addresses issues that arise in such drawings and accompanying talk (e.g., errors or interesting mathematical thinking). Middle school teachers can use students’ prior knowledge as a basis for building new understandings (Webb, Boswinkel, and Dekker 2008). Students can deepen their mathematical understandings by being led to make connections between their own representations that are “often grounded in … experiences with real or imagined contexts” (p. 112) and new, less contextually bound representations purposefully introduced by the teacher. Examples are included throughout this publication as to how grade 6 teachers can help their students make the transition from concrete and numerical representations to algebraic reasoning, generalization, and abstract representations.

An Important Grade 6 Issue: Developing Fluency with Multiplication and Division of Fractions and Decimals

The grade 6 Focal Point on multiplication and division of fractions and decimals is a crucial one. A solid understanding of the operations and of fractions and decimals as numbers will provide students with the foundation they need to be able to identify the appropriate times to use multiplication and division when fractions and decimals are involved. The related section of this publication presents strategies that help students learn the computational procedures in meaningful contexts. Students should be encouraged to connect their understandings of the contexts involving whole numbers that call for multiplication and division to representations for multiplication and division situations that involve fractions and decimals. Students should then be given many opportunities to use these representations to build meaning for generalizable procedures for multiplying and dividing with fractions and decimals.

Multiplication and division of fractions and decimals permeate the mathematics that students learn in grade 6 and in later grades. For example, students use these computational procedures in solving equations and working with ratios and proportions. Therefore, much attention should be dedicated to building meaning for the procedures for multiplying and dividing fractions and decimals, with the eventual goal being the ability to apply them effectively in problem-solving situations.

The three grade 6 Focal Points and their Connections (NCTM 2006, p. 18) are reproduced on the following page.

Curriculum Focal Points and Connections for Grade 6

The set of three Curriculum Focal Points and related Connections for mathematics in grade 6 follow. These topics are recommended content emphases for this grade level. It is essential that these Focal Points be addressed in contexts that promote problem solving, reasoning, and designing and analyzing representations.

Grade 6 Curriculum Focal Points	Connections to the Focal Points
Number and Operations: **Developing an understanding of and fluency with multiplication and division of fractions and decimals** Students use the meanings of fractions, multiplication and division, and the inverse relationship between multiplication and division to make sense of procedures for multiplying and dividing fractions and explain why they work. They use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain the procedures for multiplying and dividing decimals. Students use common procedures to multiply and divide fractions and decimals efficiently and accurately. They multiply and divide fractions and decimals to solve problems, including multistep problems and problems involving measurement.	***Number and Operations:*** Students' work in dividing fractions shows them that they can express the result of dividing two whole numbers as a fraction (viewed as parts of a whole). Students then extend their work in grade 5 with division of whole numbers to give mixed number and decimal solutions to division problems with whole numbers. They recognize that ratio tables not only derive from rows in the multiplication table but also connect with equivalent fractions. Students distinguish multiplicative comparisons from additive comparisons. ***Algebra:*** Students use the commutative, associative, and distributive properties to show that two expressions are equivalent. They also illustrate properties of operations by showing that two expressions are equivalent in a given context (e.g., determining the area in two different ways for a rectangle whose dimensions are $x + 3$ by 5). Sequences, including those that arise in the context of finding possible rules for patterns of figures or stacks of objects, provide opportunities for students to develop formulas. ***Measurement and Geometry:*** Problems that involve areas and volumes, calling on students to find areas or volumes from lengths or to find lengths from volumes or areas and lengths, are especially appropriate. These problems extend the students' work in grade 5 on area and volume and provide a context for applying new work with equations.
Number and Operations: **Connecting ratio and rate to multiplication and division** Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost \$3.75 and all items are the same price, then I can find the cost of 12 items by first dividing \$3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.	
Algebra: **Writing, interpreting, and using mathematical expressions and equations** Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. They understand that variables represent numbers whose exact values are not yet specified, and they use variables appropriately. Students understand that expressions in different forms can be equivalent, and they can rewrite an expression to represent a quantity in a different way (e.g., to make it more compact or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. They solve simple one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation. They construct and analyze tables (e.g., to show quantities that are in equivalent ratios), and they use equations to describe simple relationships (such as $3x = y$) shown in a table.	

Reprinted from *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (Reston, Va.: NCTM, 2006, p. 18).

2 Focusing on Multiplication and Division of Fractions and Decimals

In grade 6, students develop an understanding of, and fluency with, multiplication and division of fractions and decimals. The focus is on understanding how to represent multiplication and division of both fractions and decimals to develop an understanding of the results of these operations when fractions and decimals are involved. The eventual goal of this Focal Point is for students to be able to move flexibly between fractions and decimals and to become proficient with efficient, generalizable procedures to multiply and divide with fractions and decimals to solve problems.

Instructional Progression for Multiplication and Division of Fractions and Decimals

The focus on multiplication and division of fractions and decimals in grade 6 is supported by a progression of related mathematical ideas before and after grade 6, as shown in table 2.1. To give perspective to the grade 6 work, we first discuss some of the important ideas that students focused on before grade 6 that prepare them for multiplication and division in grade 6. At the end of the detailed discussion of this grade 6 Focal Point, we present examples of how students will use the multiplication and division understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books from NCTM, for example, *Focus in Grade 3*, *Focus in Grade 4*, *Focus in Grade 5*, *Focus in Grade 7*, and *Focus in Grade 8*.

Table 2.1 represents an instructional progression for the conceptual understanding of multiplication and division of fractions and decimals before, during, and after grade 6.

Early Foundations in Multiplication and Division of Fractions and Decimals

Before entering the grade 6 classroom, students are expected to learn many concepts and skills that they can use to understand their work in multiplying and dividing fractions and decimals. Students have multiplied and divided with whole numbers, and they have learned about the mathematical relationship between multiplication and division. Students have also worked with fractions and decimals and learned about the equivalence relationships between fractions and decimals. All these skills work together to facilitate students' understanding of the concepts in this Focal Point.

Table 2.1

Grade 6: Focusing on Multiplication and Division of Fractions and Decimals—Instructional Progression for Multiplication and Division of Fractions and Decimals

Before Grade 6	Grade 6	After Grade 6
Students develop an understanding of, and fluency with, multiplication and division of whole numbers. Students develop an understanding of fractions and fraction equivalence. Students develop an understanding of decimals, including the connection between fractions and decimals. Students develop an understanding of the area of a rectangle as the product of its length and width.	Students model multiplication with fractions and connect models of multiplication with fractions to procedures for multiplying fractions. Students model division with fractions and connect models of division of fractions to procedures for dividing fractions. Students use fractions, mixed numbers, and decimals to represent quotients in division of whole numbers.* Students recognize and use the place-value patterns in multiplying and dividing finite decimals by powers of 10. Students use place value and their understanding of multiplication of fractions to justify procedures for multiplying finite decimals. Students use place value and their understanding of representing quotients as fractions to justify procedures for dividing decimals. Students develop efficient, accurate, and generalizable methods for multiplying and dividing fractions and decimals. Students use multiplication and division of fractions and decimals to solve problems.	Students use models of negative numbers to represent and justify rules for adding, subtracting, multiplying, and dividing with negative numbers. Students develop efficient, accurate, and generalizable methods for operating with negative numbers. Students recognize fractions, percents, and certain decimals as ways of representing rational numbers and can convert among fractions, decimals, and percents. Students can explain which fractions correspond to terminating decimals. Students use linear equations in one variable and rational numbers to solve word problems. Students use exponents and scientific notation to describe very large and very small numbers.** Students use square roots when they apply the Pythagorean theorem.**

*Appears in the Grade 6 Connections to the Focal Points (NCTM 2006).
** Appears in the Grade 8 Connections to the Focal Points (NCTM 2006).

Early Foundations in Whole-Number Multiplication and Division

In previous grades, students are expected to develop an understanding of, and fluency with, multiplication and division of whole numbers that form a crucial foundation for understanding multiplication and division of fractions and decimals. In grade 4, students use place value, a knowledge of basic facts, and patterns to find products involving powers of 10 (10, 100, 1,000, and so on) and multiples of those powers (such as 10, 20, 30, 100, 200, 300, 1,000, 2,000, 3,000). In grade 5, students build on patterns for multiplying by powers of 10 to learn and apply patterns for dividing multiples of powers of 10, for example, 6 ÷ 3 = 2, 60 ÷ 3 = 20, and 600 ÷ 3 = 200. The tools and concepts that help prepare students for multiplication and division of whole numbers, such as understanding the models for multiplication (i.e., equal-sized groups, arrays, area models, and equal intervals on the number line), developing a quick recall of multiplication and related division facts, understanding place value, and understanding the properties of operations (the distributive property in particular), also prepare them for multiplication and division of fractions and decimals.

In grades 3 and 4, students learn about two interpretations of division: sharing or partitive (how many in each group?) and measurement or quotitive (how many groups?). They use their understanding of these division situations as they explore what it means to divide fractions and decimals. Students also use arrays and area models to help them reason about multiplication and division and can use these models to foster new learning when multiplying fractions and decimals.

Another foundational concept that students are expected to develop in earlier grades is an understanding of the number of square units in the area of a rectangle as the product of the numbers of units in its length and width. In grade 4, students explore area as the number of same-sized square units that cover a two-dimensional shape without gaps or overlaps. They use area models to represent multiplication and to develop *length times width* as the expression for the area of a rectangle.

Early Foundations in Understanding Fractions and Decimals

The understandings of fractions, fraction equivalence, and decimals that students are expected to develop in earlier grades are also foundational principles that are crucial to students' understanding of multiplication and division of fractions and decimals. In grade 3, students compare unit fractions

$$\frac{1}{n}$$

of the same-sized whole by observing that the greater the denominator, the smaller the amount represented by the unit fraction. They represent a fraction as part of a whole, part of a set, a point on a number line, or a distance on a number line. They order fractions by using models, benchmark fractions, or common numerators or denominators. They also identify equivalent fractions by using models, including the number line. In grade 5, students reinforce skills with equivalent fractions when they add and subtract fractions that have unlike denominators.

Before grade 6, students are expected to develop an understanding of decimals and the connection between fractions and decimals. In grade 4, students extend the base-ten system to include decimal notation. They use decimals to represent fractions whose denominators are powers of 10. They write equivalent fractions and decimals, for example,

$$\frac{4}{10} = \frac{40}{100} = 0.4 = 0.40,$$

and connect them to models, such as hundreds-grids and points on a number line. They use place-value notation and their understanding of fractions to compare and order decimals. The understanding of place value that students use in adding and subtracting decimals in grade 5 will be applied in grade 6 as they develop procedures for multiplying and dividing decimals.

Building an Understanding of, and Fluency with, Multiplication and Division of Fractions and Decimals

Although the ideas are presented in a particular order in this Focal Point, it is not expected that these topics will be addressed in this order in the classroom. Teachers should always look for ways to interconnect the concepts. Students themselves should see and offer descriptions of how the ideas are related. These kinds of relationships should be sought after, identified, and promoted.

In a classroom where instruction is based on a focused curriculum with emphasis on connections among ideas, it is often desirable to present for students the need to use a new mathematical idea in a context to solve a problem. For example, the situations shown in figure 2.1 could be presented when introducing multiplication with fractions.

1. A cake recipe calls for 2/3 cup of flour. How much flour is needed for 4 cakes?
2. Eva takes 4 pints of water for a hike up a mountain trail and back. She plans to drink 2/3 of the water on the way up. How many pints does Eva plan to drink on the way up?

Fig. 2.1. Situations that can help facilitate students' thinking about multiplication of fractions

When discussing problem 1, students should be guided to think, "When I had 4 groups with 3 in each group, I used multiplication, 4 × 3. If I apply the same meaning of multiplication here, then 4 groups with 2/3 in each group would be

$$4 \times \frac{2}{3}."$$

When discussing problem 2, students should think, "If 5 groups of 4 is 5 × 4, then 2/3 of a group of 4 would be

$$\frac{2}{3} \times 4."$$

The differences and similarities between these problems and problems like them should be explored so that students begin to see the relationships of multiplication of fractions and decimals to multiplication of whole numbers. In addition, students must distinguish between the situations associated with multiplication of fractions and decimals and those associated with division of fractions and decimals.

Fractions

As students begin to connect their previous knowledge with their developing understanding of multiplying and dividing fractions, the representations that students use become very important. In earlier grades, students have encountered contexts for, and representations of, fractions and multiplication and division with whole numbers. However, these contexts and representations may not necessarily be effective for developing understanding of multiplication and division of fractions. Teachers should provide opportunities for students first to use the models with which they are familiar and then instruct students how to connect their models to those that can lead to more abstract representations and understanding of the corresponding concepts and procedures.

Multiplying fractions

If students have previously learned about the commutative property of multiplication, they know that the expressions

$$4 \times \frac{2}{3}$$

and

$$\frac{2}{3} \times 4$$

have the same value. However, they need to understand that the expressions can have different interpretations, as indicated in the examples in figure 2.1. For example, the expression

$$4 \times \frac{2}{3}$$

can be interpreted as 4 groups of 2/3 or 4 times 2/3 (4 times as big, 4 times as far, 4 times as heavy, and so on.) In contrast,

$$\frac{2}{3} \times 4$$

can be interpreted as 2/3 of a group of 4 or 2/3 times 4. The chart in figure 2.2 summarizes some important aspects of multiplication. Note that both rectangular area models and number lines are shown in the chart. The rectangular area model is helpful in making a visual connection between multiplication of whole numbers and multiplication of fractions. However, students can also benefit from using the number line because it enables them to connect the number-line model of division of fractions to the number-line model of multiplication of fractions.

Modeling multiplication with fractions

The diagrams shown in figure 2.2 that use rectangular areas are examples of models used for multiplication. Different students are likely to construct different models for the same situation. For example, suppose students are presented the Recipe Problem:

A cake recipe calls for 2/3 cup of flour. How much flour is needed for 4 cakes?

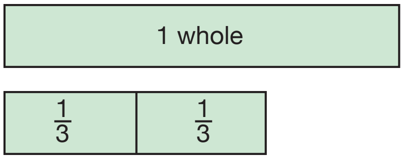
Two different models that students might use are presented in the following classroom discussion.

Reflect As You Read

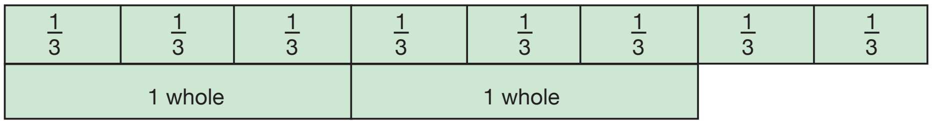
Before continuing, draw at least one, and if possible two, models for the problem. If you can think of only one way to illustrate this problem, be sure to fully examine the representations presented in the following classroom discussion.

Teacher: As I was walking around, I saw that many of you used different models to solve the problem. David, can you show your model and explain how you used it to solve the problem?

David: I used fraction bars to show 2 thirds for the amount of flour in each cake.



Then I made 3 more groups of 2 thirds bars to have a total of 4 groups of 2 thirds. That's 8 thirds altogether. Then, I used whole bars to see that 8 thirds is equal to 2 wholes and 2 thirds, or 2 and 2/3.



So

$$4 \times \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3},$$

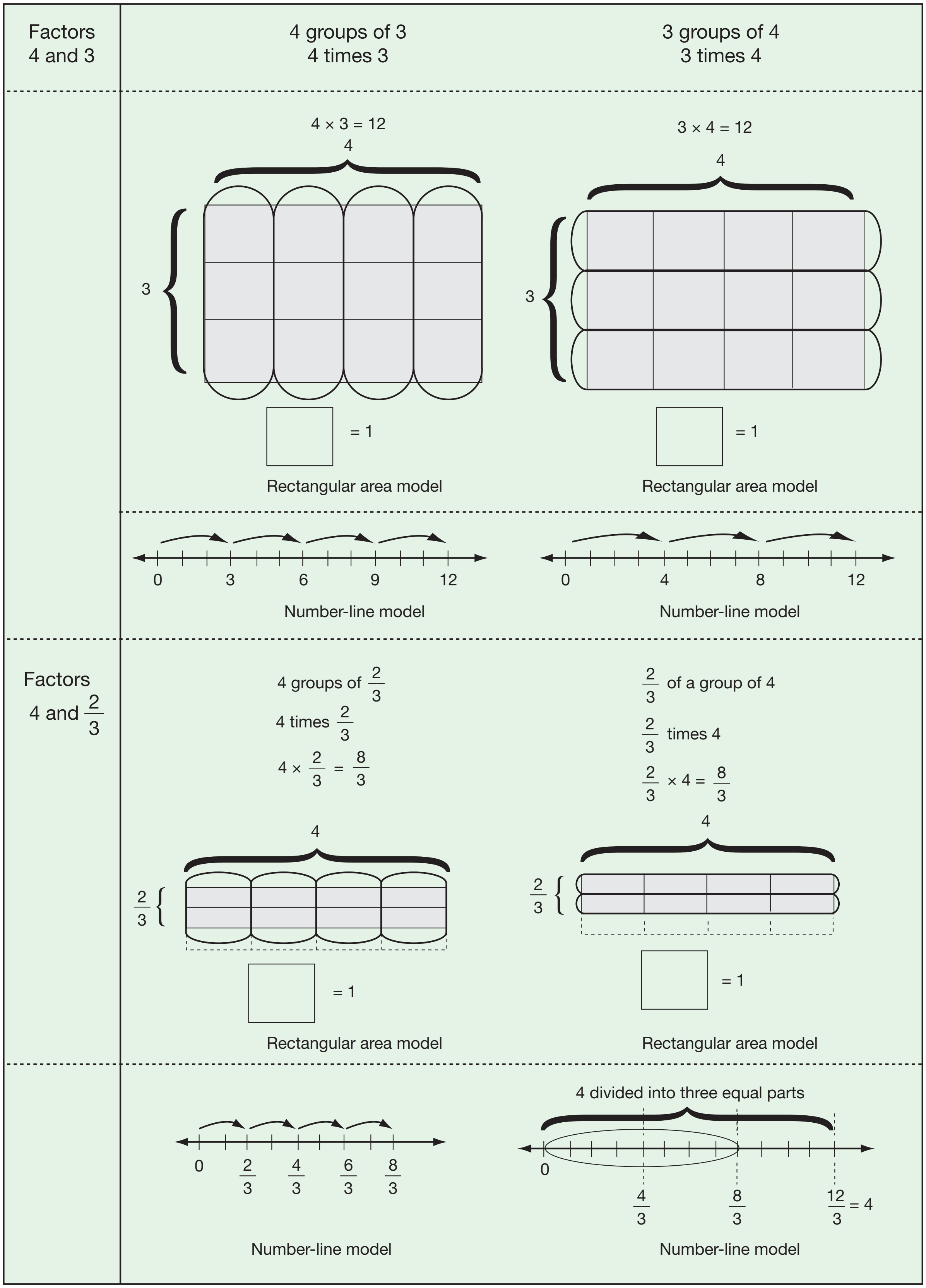


Fig. 2.2. Chart summarizing important concepts about multiplication

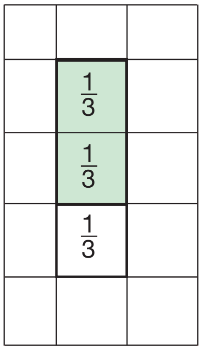
and

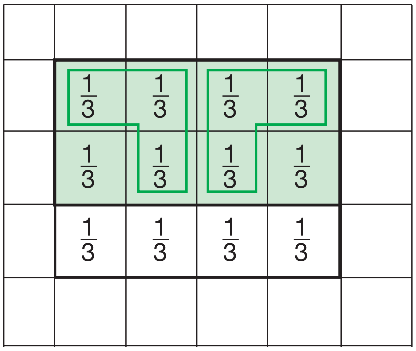
$$2\frac{2}{3}$$

cups of flour are needed for 4 cakes.

Anne: I did the problem another way. I drew a rectangle on grid paper to represent one whole. I divided it into 3 equal parts to represent thirds and then shaded 2 parts to represent 2/3. I repeated the diagram until I had 4 groups of 2/3. There are 3 thirds in one whole, so I circled groups of 3 thirds to show wholes. There were 2 wholes and 2 thirds left over, so

$$4\times\frac{2}{3}=2\frac{2}{3}.$$





Reflect As You Read

Think about Anne's and David's models. Then answer these questions:

- Can Anne's and David's models easily be used to model multiplication problems involving two fractions or mixed numbers?
- What are the limitations of these models?
- What model might be able to be applied to a greater variety of problem types and more readily lead students to a generalizable symbolic representation?

The students' models in the preceding discussion are representations for solving the problem on the basis of their previous work with fractions and whole-number multiplication. However, teachers can move students from their use of these representations to a model that can be applied to a greater variety of problem types and more complex problem types and that can lead to understanding of a generalizable symbolic representation. Through discussions such as the one that follows, teachers can start to transition students to a grid model that shows products as areas. For example, the product of 4 and 5 is shown in figure 2.3 as 20 square units, the number of square units in the area of a 4-by-5 rectangle. The continuation of the previous classroom discussion shows how teachers can transition students to the grid model.

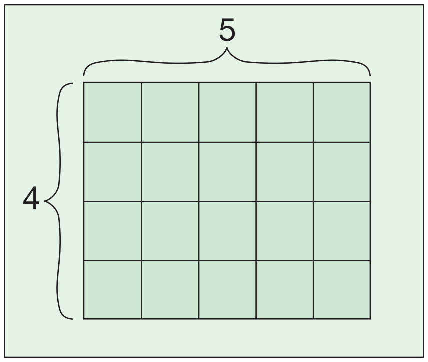


Fig. 2.3. A 4-by-5 grid model representing 4 × 5 = 20

Reflect As You Read

Before continuing, draw the grid models for

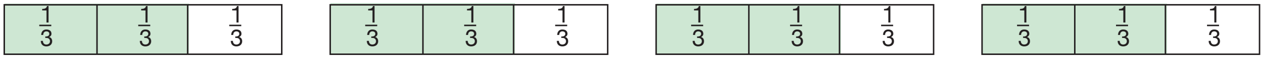
$$\frac{2}{3}\times\frac{3}{5}$$

and

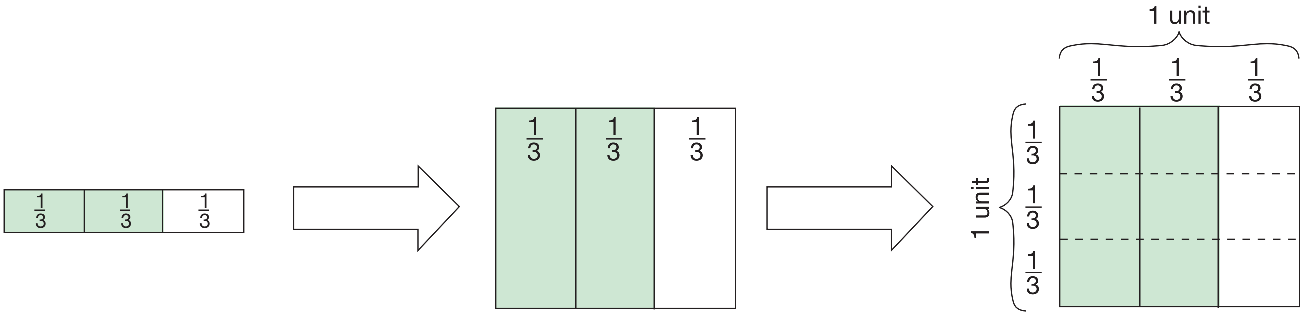
$$2\frac{2}{3}\times3\frac{1}{2}$$

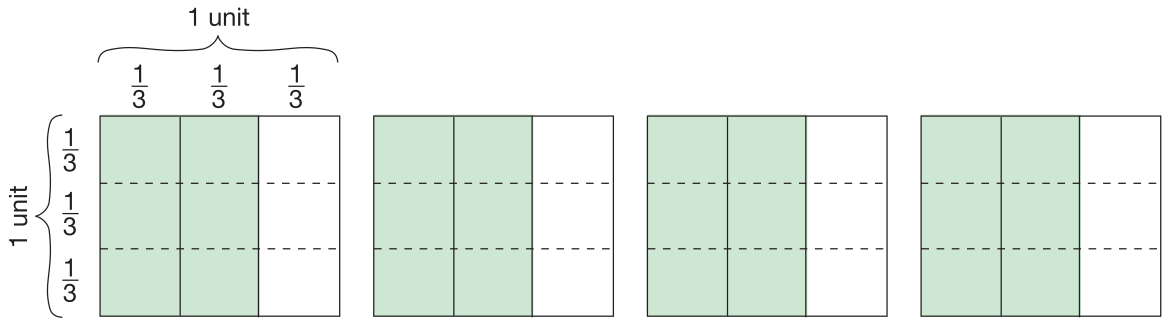
Then proceed with the classroom discussion.

Teacher: We can build on David's model by making a drawing to show how his 2 thirds fraction bars are part of a whole.



Then we can redefine the wholes to be unit squares. That is, we can make each whole 1 unit long and 1 unit wide, or 3 thirds long and 3 thirds wide.





For each unit square, the shaded portion also represents 2/3, and we still have 4 of them. Now we can stack the unit squares to make a shaded rectangle that is 4 units by 2/3 unit.

The teacher would help students make connections between David's model and the grid diagram shown in figure 2.4B.

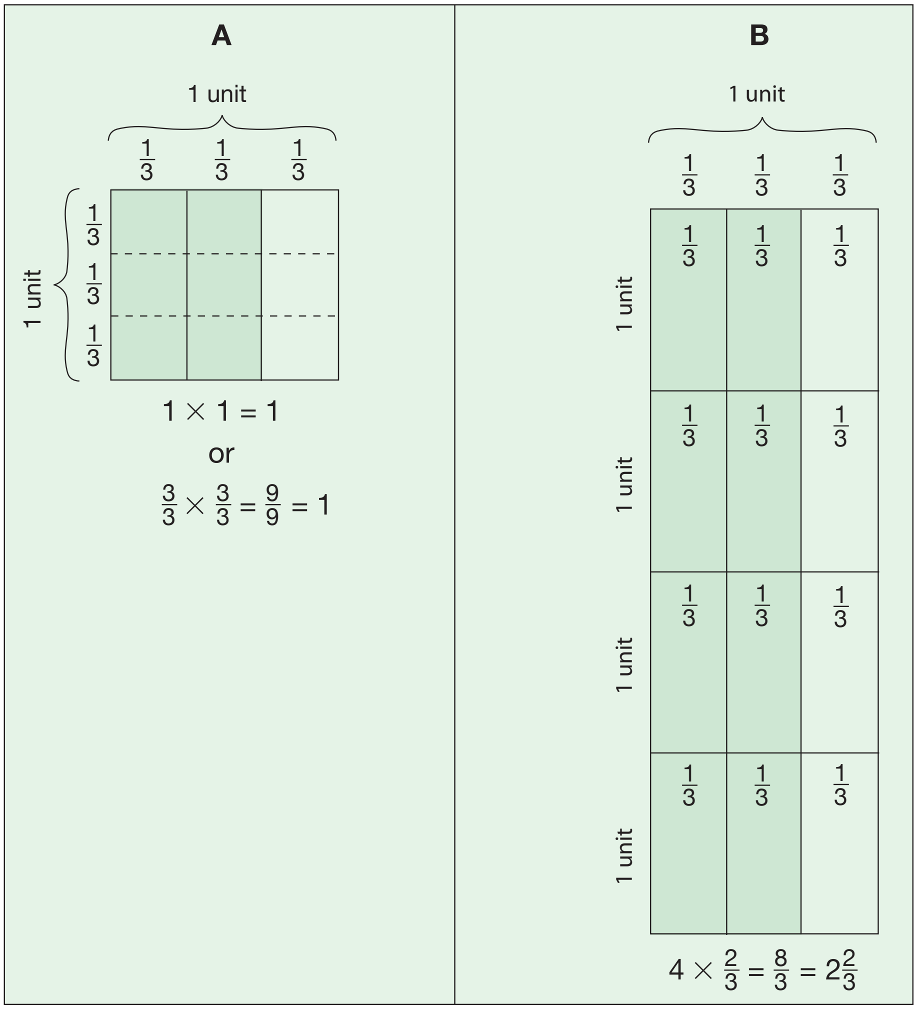


Fig. 2.4. Grid models

Teacher: So we have helped David revise his model to show a grid that is 4 units high and 2/3 units wide. The area of this grid also shows that

$$4 \times \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}.$$

Teacher: Now let's look at Anne's model. Anne, can you redraw your model to make it a unit square like David's?

The teacher would lead Anne to the diagram shown in figure 2.5.

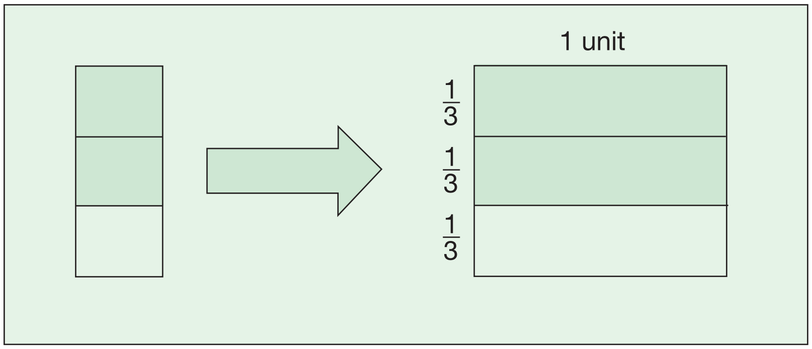


Fig. 2.5. A 2/3-by-1 grid model representing 2/3

Teacher: Anne, can you show how to put together the unit squares to make a shaded rectangle that is 2/3 unit by 4 units?

The teacher would help make the connections between Anne's model and the grid diagram as shown in figure 2.3. Now Anne has revised her model to show a grid like David's grid, as shown in figure 2.6.

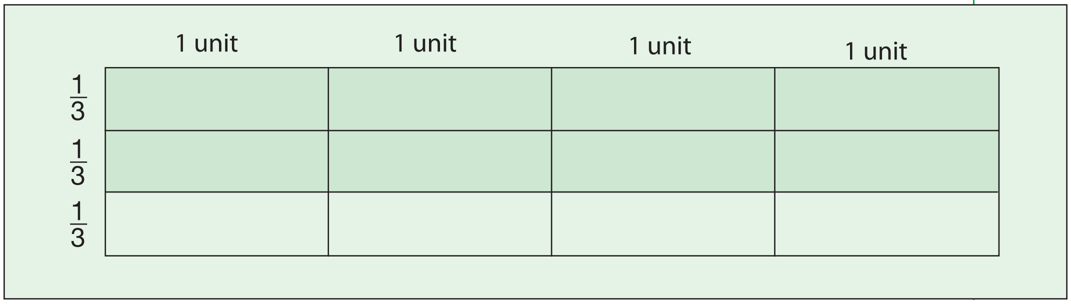


Fig. 2.6. Model showing a grid that is 2/3 units by 4 units

Teacher: The area of this grid also shows that the product of 4 and 2/3 is 8/3 or 2 2/3. Both grid models show congruent shaded rectangles with dimensions 4 units by 2/3 unit. Grid models are useful because they can represent multiplication of whole numbers, fractions, mixed numbers, or decimals.

Now students should be ready to learn how to use grid models for multiplication of fractions or mixed numbers. Two examples of how to use grid models to model fraction multiplication are shown in figure 2.7.

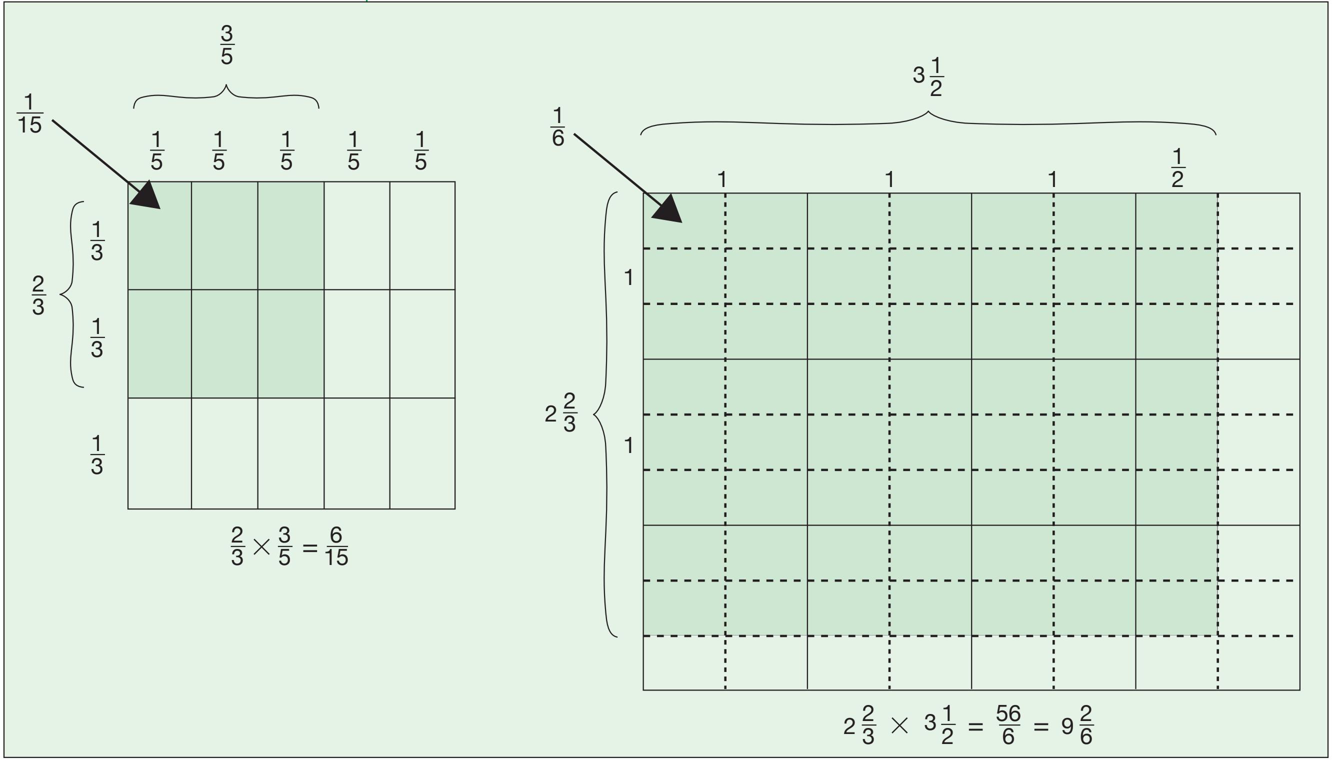


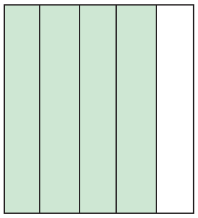
Fig. 2.7. Examples of how to use grid models to represent multiplication with fractions

Procedures for multiplying fractions

As students conduct in-depth analysis of the meaning of a multiplication grid model, their understanding of the multiplication procedure will also deepen. Such an analysis can be elicited as shown in the following discussion of how the grid model can be used to represent

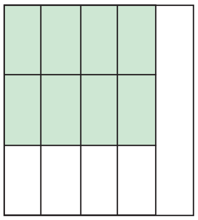
$$\frac{2}{3} \times \frac{4}{5}.$$

Teacher: This model shows one whole divided into 5 equal parts with 4 parts shaded. What does the model represent? [4/5]

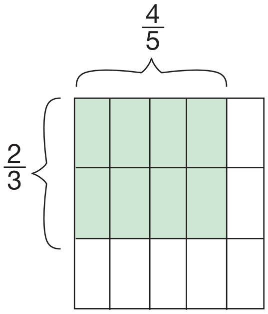


Teacher: This model shows 4/5 divided into 3 equal parts with 2 parts shaded. What does this model represent?

$$\left[\frac{2}{3} \text{ of } \frac{4}{5}, \text{ or } \frac{2}{3}\times\frac{4}{5}\right]$$



Teacher: This model shows 2/3 of 4/5 so that it can be compared to the entire one whole.



How many congruent sections are in the entire one whole, and how many are shaded? [There are 15 sections in the entire one whole, and 8 are shaded.] What part of the entire one whole is 2/3 of 4/5? [8/15] So what is

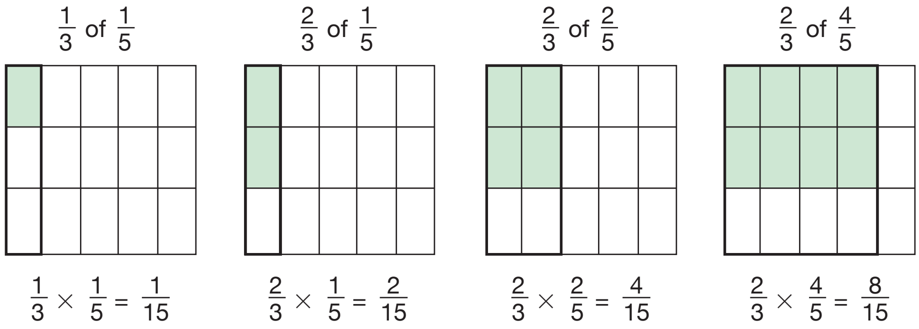
$$\frac{2}{3}\times\frac{4}{5}?\ \left[\frac{8}{15}\right]$$

After students have had experience connecting their own models to the grid model to represent many different kinds of fraction multiplication problems, they should move to connecting those models to the formal procedure for multiplying fractions. One way to do so is to use the grid models to identify patterns. The following classroom discussion illustrates how grids and patterns can help students connect the understanding of fractions that they have developed through modeling to the development of a meaningful procedure for multiplying fractions.

Teacher: Let's use grids to model

$$\frac{1}{3}\times\frac{1}{5}.$$

Now let's change the numerators of the fractions and model each new problem.



Notice that to represent both thirds and fifths in your grid, you can first divide the figure vertically into 5 equal parts to identify 1/5 of the whole, and then you can divide the 1/5 section horizontally into 3 equal parts to find 1/3 of 1/5. If you continue to divide the whole rectangle into fifths, how many parts are there in your grid in all? [15] So your grid shows 15ths, because cutting all of the fifths into 3 pieces each gives 15 pieces in the whole. So in the part you shaded, you are shading a certain number of 15ths. Now look at your equations. What are the denominators in each factor? [3 and 5] What is the denominator in each product? [15] How can you use the denominators in the factors to get the denominator in the product? [Multiply them.] Why does that work? [Because we are cutting thirds into five parts or fifths into three parts.] Now look at how you shaded your grids. The numerator of the first factor tells you how many sections end up being shaded from the thirds pieces. The numerator of the second factor tells you how many vertical sections end up being shaded from the fifths pieces. So you end up shading a section of the grid whose dimensions are the first factor's numerator and the second factor's numerator. Look at the numerators in each problem and the numerator in the product. How can you use the numerators in the factors to get the numerator in the product? [Multiply them.]

After students have had ample exposure to these types of experiences, they should begin to have a sense of the relative size of a product and be able to understand and correctly explain why the procedure of multiplying fractions involves multiplying the numerators to get the numerator in the product and multiplying the denominators to get the denominator in the product.

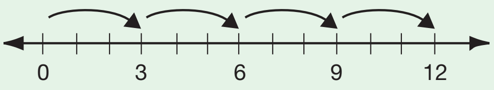
Dividing fractions

Division has two common interpretations—measurement (or quotitive) and sharing (or partitive). (See *Focus in Grade 3* [NCTM 2009, pp. 19–21].) Each interpretation has its associated characteristic language. Patterns in language used with dividing whole numbers can help develop understanding for division examples that involve fractions. The chart in figure 2.8 describes three different division examples, first using the quotitive interpretation and then using the partitive interpretation. A number line is used to represent each example. The number line can be a very efficient model for representing division. If students have used the number line to model fraction multiplication, they will have the experience necessary to connect the number line model of division of fractions back to the number line model of multiplication of fractions.

<table>
<tr><th></th><th>Quotitive (or Measurement) Division</th><th>Partitive (or Sharing) Division</th></tr>
<tr><td>12 ÷ 3</td><td>

If 3 is one group, how many groups can you make with 12?

12 in groups of 3

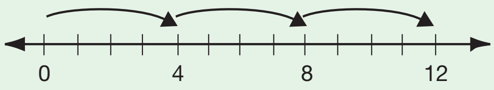


There are 4 groups of 3.

</td><td>

If 12 is 3 groups, how many are in one group?

12 is 3 groups

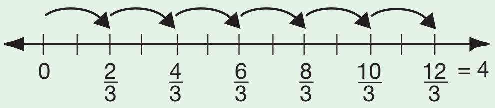


4 is one group.

</td></tr>
<tr><td>$4 \div \frac{2}{3}$</td><td>

If 2/3 is one group, how many groups can you make with 4?

4 in groups of $\frac{2}{3}$

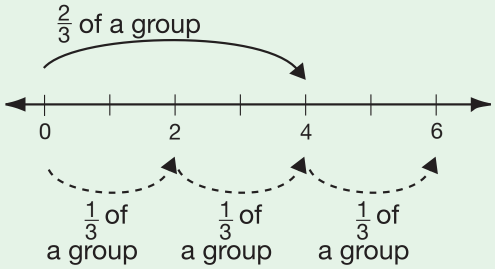


There are 6 groups of $\frac{2}{3}$.

</td><td>

If 4 is 2/3 of a group, how many are in one group?

4 is $\frac{2}{3}$ of a group.

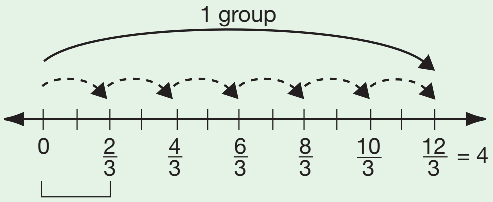


6 is one group.

</td></tr>
<tr><td>$\frac{2}{3} \div 4$</td><td>

If 4 is one group, how many groups can you make with $\frac{2}{3}$?

$\frac{2}{3}$ in groups of 4

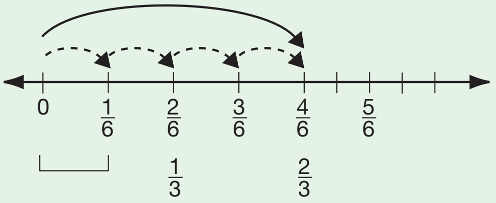


There is $\frac{1}{6}$ of a group of 4.

</td><td>

If $\frac{2}{3}$ is 4 groups, how many are in one group?

$\frac{2}{3}$ is 4 groups



$\frac{1}{6}$ is one group.

</td></tr>
</table>

Fig. 2.8. Chart showing quotitive and partitive interpretations of division

Modeling division with fractions

Division has different interpretations, and so students may use different types of models to represent different situations. Also, the same model may be used in a different way depending on the context of the problem. In other words, a student might use fraction bars to model both a quotitive and a partitive division problem, but the model will look different. As with multiplication, teachers should help students connect their prior understanding of division with whole numbers to models for representing division that can lead to formal symbolic procedures for dividing fractions.

Modeling measurement division

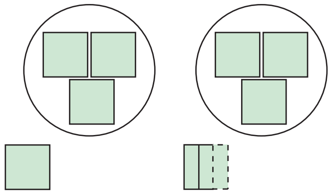
Consider this problem:

> You have 7 2/3 pounds of dog food. You feed your dog 3 pounds per week. How many weeks will the dog food last?

This is a measurement division problem because the quotient will represent the number of groups of 3 pounds. Again, as the following classroom discussion shows, students might use the set model to represent this problem. For this particular problem, however, they may realize that it is difficult to completely solve the problem using this model. It is at that point that the teacher can help students move into the use of a model that works better for representing division with fractions. With these representations introduced by the teacher and connected to their prior understandings, students can finish solving the problem and, at the same time, move one step closer to understanding the standard procedure for dividing fractions.

Teacher: Tony, can you show your model and explain how you solved the problem?

Tony: I knew that I had to split up 7 2/3 things into groups of 3, so I dealt out the objects into groups of 3 until I couldn't deal out any more. I had 2 groups of 3 pounds with 1 2/3 pounds left.

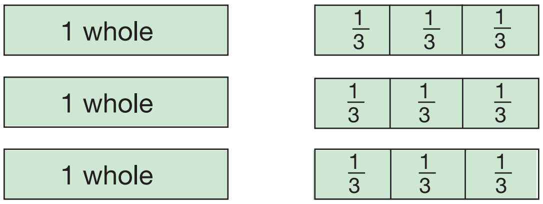


1 pound and $\frac{2}{3}$ of a pound left

I had 2 groups and a part of a group. Then I had to figure out how much 1 2/3 is of a third group of 3. That's where I got stuck. I know that the answer to the problem is 2 and some fraction, but I am not sure how to use my model to find the fraction part of my answer.

Teacher: Great job so far, Tony. Let's see how some other people thought about it. Josie, I see that you used fraction strips to solve the problem, and your solution looks the same as Tony's—you have 2 groups of 3 bars and 1 and 2/3 bars left over. Let's see if we can figure it out using the fraction bars. It takes 3 wholes to make one group, so how many thirds are in 1 group?

Josie: There are 9 thirds in 1 group of 3.



Teacher: That's right. Now use your bars to find out how many thirds are in the leftover 1 2/3.

Josie: There are 5 thirds left over.

Teacher: Okay. So how much of a whole group is this part that is left over?

Tony: I think I get it. If you need 9 pieces to make a whole group, and you only have 5 pieces, then you have 5/9 of what you need for a whole group. So from 7 and 2/3 pounds, I can make 2 and 5/9 groups of 3 pounds, and the dog food will last 2 5/9 weeks. That's a weird number of weeks!

Teacher: So, we could say that 7 2/3 pounds of dog food put into packages of 3 pounds for each week would last for 2 and 5/9 weeks. And we could write that mathematically as

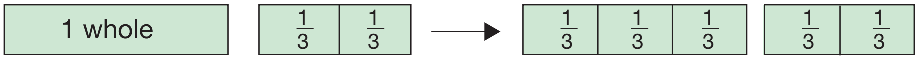
$$7\frac{2}{3} \div 3 = 2\frac{5}{9}.$$

At the appropriate time, teachers can use these discussions on previous problems to help students transition to a number-line model. The number-line diagrams shown in the subsequent discussion lend visual support for the development of understanding of division with fractions. The diagrams should accompany the discussion.

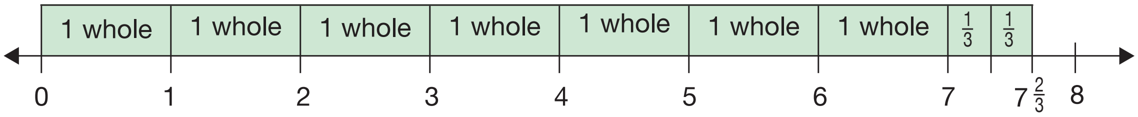
Teacher: Let's look back at the Dog Food Problem and use a number line model to show

$$7\frac{2}{3} \div 3$$

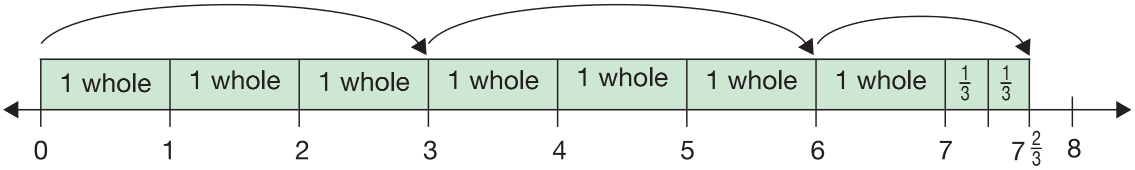
as the answer to the question "How many groups of 3 are in 7 2/3?"



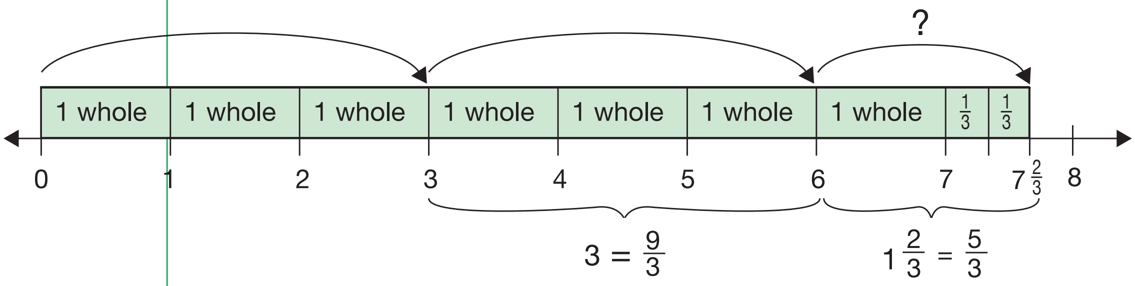
We know that we have to start at 0 and make jumps of 3 to 7 2/3. Since we know how to use fraction bars to find the answer, let's use fraction bars on our number line. So place fraction bars for 7 2/3 on a number line.



To show the groups of 3, make jumps of 3 until we can't make any more jumps of 3. We make 2 jumps of 3 and land on 6.



To get from 6 to 7 2/3, we need to make one jump of 1 2/3. But since 1 2/3 is less than 3, a jump of 1 2/3 is less than one complete jump. To find out how much of a jump a 1 2/3 jump is, we have to figure out how much 1 2/3 is of 3. Because we are working with thirds, we can divide the whole number line into thirds.



We can count thirds on the number line and discover that a full jump is 9 thirds. We can also count thirds to discover that a jump of 1 2/3 is 5 thirds. So a jump of 1 2/3 is 5/9 of a full jump. It takes 2 jumps of 3 and 5/9 of a jump of 3 to get from 0 to 7 2/3, so

$$7\frac{2}{3} \div 3 = 2\frac{5}{9}.$$

This model for division can be translated into a procedure for dividing fractions that uses common denominators. Although this procedure is not considered the traditional procedure, it is mathematically valid and easy to use if students are comfortable with finding common denominators. Students can discuss how their model translated the question “How many groups of 3 are there in 7 2/3?” into “How many groups of 9 thirds are there in 23 thirds?” Since the groups are described using the same size pieces (thirds), the question then becomes “How many groups of 9 are there in 23?” or “What is 23 divided by 9?” Symbolically, this would look like

$$7\frac{2}{3} \div 3 = \frac{23}{3} \div \frac{9}{3} = 23 \div 9 = 2\frac{5}{9}$$

(groups of 9/3 or groups of 3).

Modeling partitive division

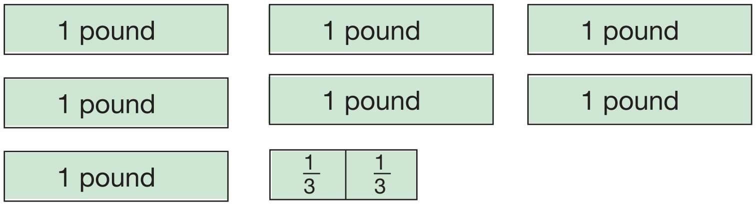
Suppose a teacher presents students with this problem:

> You have 7 2/3 pounds of almonds. You want to put them into 3 gift baskets, putting the same amount into each basket. How many pounds of almonds should you put into each basket?

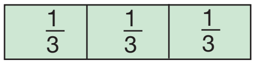
This is a partitive division problem because the quotient will represent the amount in one group. Students might use different models to represent and solve this problem. As the following classroom discussion shows, students might use a familiar model for fractions, such as fraction bars, to represent the sets involved.

Teacher: Sasha, I see you used fraction bars. Can you show your solution?

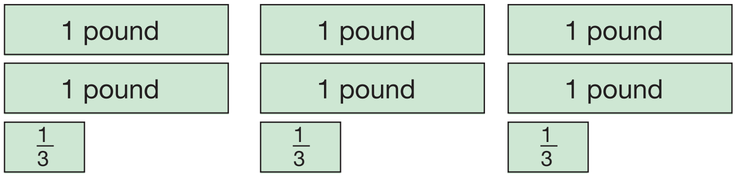
Sasha: I know that I have to share 7 2/3 pounds equally among 3 groups. I think of one fraction bar as one pound and lay out 7 pounds and 2 one-third pounds. It’s easy to see I can share 2 pounds with each of the 3 groups.

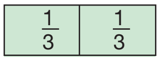


I have 1 2/3 pounds left. So I break that one pound into 3 equal one-third-pound pieces.

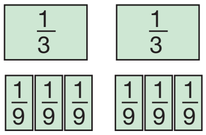


So I put 1 one-third pound in each group. And I have 2 one-third pieces left to split up.

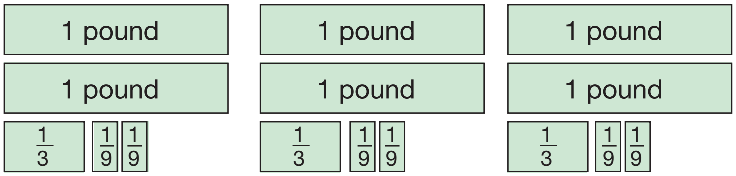




Now I can break each one-third into 3 equal pieces. One-third equals 3 one-ninths, so 2 one-third pounds equals 6 one-ninth pounds.

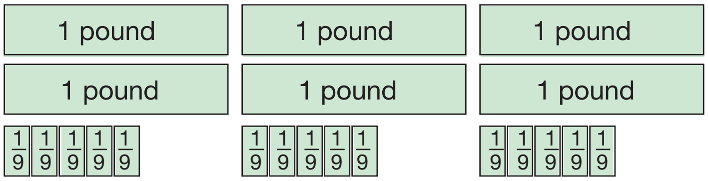


Now I can share 2 one-ninth pounds with each group.



Teacher: Very interesting, Sasha. You used fraction bars to show how to divide 7 and 2/3 pounds into 3 equal groups. How much is in one group?

Sasha: I see that I have 2 pounds plus 1/3 pound plus two 1/9 pounds. I use one-ninth fraction bars to help show that since 1 one-third is equal to 3 one-ninths, I have 5 one-ninths in all. I have 2 pounds and 5 one-ninth pounds in each group.



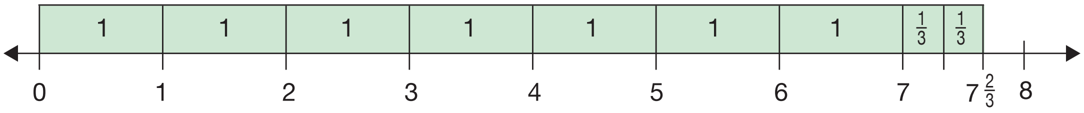
So

$$7\frac{2}{3} \div 3 = 2\frac{5}{9},$$

and 2 5/9 pounds of almonds should go in each basket.

At the appropriate time, a classroom discussion like the one that follows can aid students in making the transition from using a fraction-bar model to using a number-line model.

Teacher: Let's use a number line to show how much will be in each group if you divide 7 2/3 into 3 equal groups. However, let's also use fraction bars to help understand how we use the number-line model. To find the number in each of 3 groups, we need to divide the distance from 0 to 7 2/3 into 3 equal sections and find out the size of each section. Notice how I have combined fraction bars and the number line to show 7 2/3.

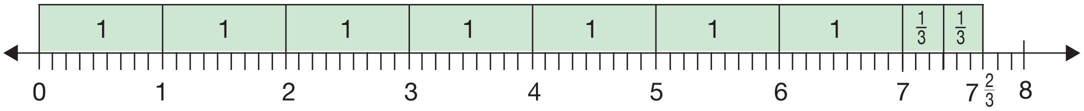


As we look at the number line, we would like to divide the interval from 0 to 7 2/3 into 3 equal parts. To do so, we need to mark this part of the number line into equal parts where the total number of parts is divisible by 3. Let's start with breaking each one-unit interval into 3 one-third intervals. Note that

$$7\frac{2}{3} = 7 + \frac{2}{3} = \frac{21}{3} + \frac{2}{3} = \frac{23}{3}.$$

But 23, the number of intervals, cannot be divided evenly by 3, so using intervals of length one-third will not work. We know that

$$\frac{1}{3} = \frac{2}{6},$$



so let's try sixths:

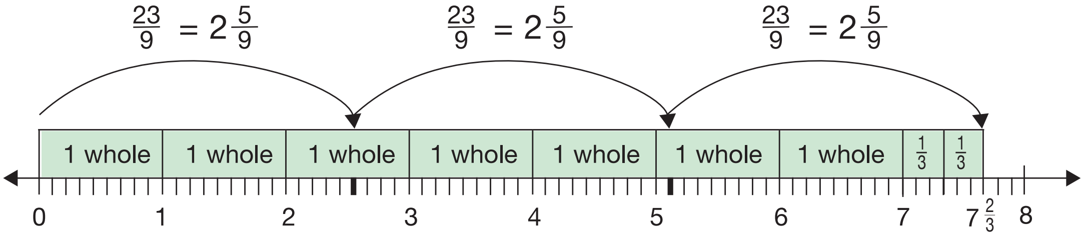
$$\frac{23}{3} = \frac{23}{3} \times \frac{2}{2} = \frac{46}{6}.$$

But 46 cannot be divided evenly by 3, so this will not work either. Let's try intervals of length one-ninth:

$$\frac{23}{3}=\frac{23}{3}\times\frac{3}{3}=\frac{69}{9},$$

and 69 can be divided by 3, so this will work. So we divide our number line into intervals of length one-ninth. There will be 69 intervals between 0 and 7 2/3.

Now because 69 ÷ 3 = 23, we want 23 one-ninth intervals in each of 3 parts.



So since each equal part has 23 one-ninth intervals and

$$\frac{23}{9}=2\frac{5}{9},$$

we conclude that

$$7\frac{2}{3}\div 3=2\frac{5}{9}.$$

Through multiple experiences with using the number line to model partitive division, students may gain a better understanding of the "invert and multiply" division procedure that is shown in the examples in figure 2.9.

Reflect As You Read

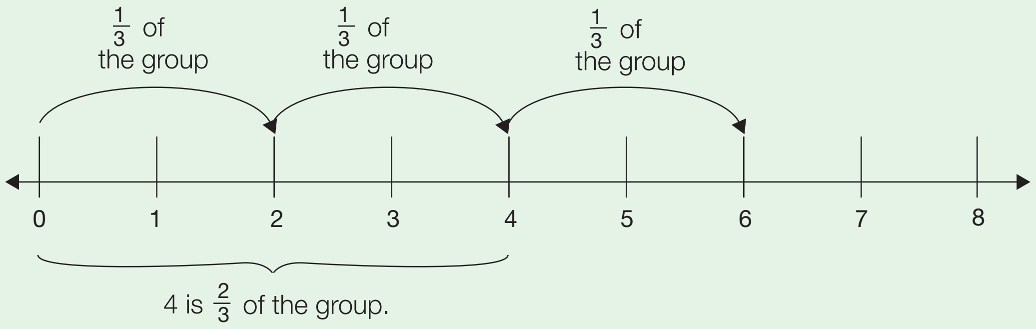
Before continuing, draw number lines to solve these problems:

1. Julie is making trail mix. She is told that 4 cups of raisins make 2/3 of a batch of trail mix. How many cups of raisins are in 1 batch of trail mix?
2. Matthew is driving to his grandmother's house. He is told that when he has driven 5 miles, he is 3/4 of the way to his grandmother's house. How many miles is it to Matthew's grandmother's house?

Then proceed by reading the examples and their solutions.

Example 1: Julie is making trail mix. She is told that 4 cups of raisins make 2/3 of a batch of trail mix. How many cups of raisins are in 1 batch of trail mix?

Think: If 4 cups of raisins makes 2/3 of a batch of trail mix, and if *b* represents a whole batch, then 2/3 × *b* = 4. Thus, the answer to our problem can be obtained by dividing 4 by the fraction 2/3. One approach to finding the answer is to think that the 4 cups total 2 "parts" of a batch where there are 3 "parts" for the entire batch. We use a number line to help visualize our answer.



To find the whole group, make 3 jumps of 2, landing at 6. So

$$4 \div \frac{2}{3} = 6.$$

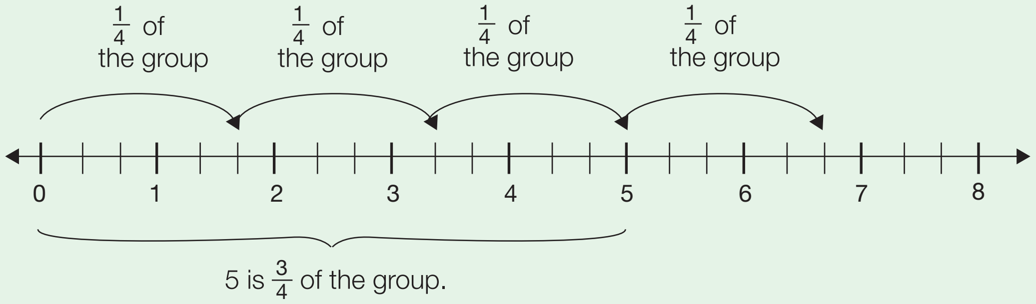
There are 6 cups of raisins in a batch of trail mix.

This makes sense because

$$\frac{2}{3} \times 6 = \frac{2}{3} \times \frac{6}{1} = \frac{12}{3} = 4.$$

Example 2: Matthew is driving to his grandmother's house. He is told that when he has driven 5 miles, he is 3/4 of the way to his grandmother's house. How many miles is it to Matthew's grandmother's house?

Think: If 5 miles is 3/4 of the trip, and if *d* represents the total distance to Grandma's, then 3/4 × *d* = 5 and so *d* = 5 ÷ 3/4. Thus, the answer to our problem can be obtained by dividing 5 by the fraction 3/4. One approach to finding the answer is to think that the 5 miles totals 3 "parts" of the trip where there are four "parts" for the entire trip. We use a number line to help visualize our answer.



(*Continued*)

If we think of each of the 5 miles as being broken into 3 intervals, we get 15 intervals, each of which is equal to $\frac{1}{3}$ mile. We break the 15 intervals into three parts (or groups) with 5 intervals in each part.

Thus, each part has length $5\times\frac{1}{3}=\frac{5}{3}$ miles and represents 1/4 of the whole trip. Next we multiply the length of an interval ($\frac{1}{4}$ of the trip) by 4 to get the distance of the total trip: $d=5\times\frac{1}{3}\times 4=5\times\frac{4}{3}$.

Note that to divide by $\frac{3}{4}$ we multiplied by $\frac{4}{3}$, that is, $d=5\div\frac{3}{4}=5\times\frac{4}{3}$.

To find the whole group, make 4 jumps of $\frac{1}{4}$, landing at $\frac{20}{3}=6\frac{2}{3}$.

So $5\div\frac{3}{4}=6\frac{2}{3}$.

It is $6\frac{2}{3}$ miles to Matthew’s grandmother’s house.

This makes sense because $\frac{3}{4}\times 6\frac{2}{3}=\frac{3}{4}\times\frac{20}{3}=\frac{60}{12}=5$.

Fig. 2.9. Examples to help students understand the “invert and multiply” procedure for dividing fractions

Procedures for dividing fractions

Through the types of examples in the previous section, the teacher can help students see that all of the following are equivalent procedures for dividing by a fraction.

- Divide by the numerator of the divisor, and then multiply by the denominator of the divisor.

$$(5\div\frac{3}{4}\rightarrow 5\times\frac{1}{3}=\frac{5}{3}$$

and

$$4\times\frac{5}{3}=\frac{20}{3}=6\frac{2}{3})$$

- Invert the divisor and multiply.
- Multiply by the reciprocal of the divisor.

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} 5\div\frac{3}{4}=5\times\frac{4}{3}=\frac{20}{3}=6\frac{2}{3} \\ \text{or} \\ a\div\frac{b}{c}=a\times\frac{c}{b} \end{array}$$

Only after students have gained the conceptual understanding of what it means to divide by a fraction should they begin to use the “invert the divisor and multiply” procedure when solving division problems without the number-line model. These examples should include a variety of types of problems and exercises, such as those shown in figure 2.10. Students should realize that writing the mixed numbers as improper fractions before dividing will facilitate the computation.

$$\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}$$

$$2\frac{1}{2} \div \frac{1}{3} = \frac{5}{2} \div \frac{1}{3} = \frac{5}{2} \times \frac{3}{1} = \frac{5 \times 3}{2 \times 1} = \frac{15}{2} = 7\frac{1}{2}$$

$$1\frac{3}{4} \div 2\frac{1}{5} = \frac{7}{4} \div \frac{11}{5} = \frac{7}{4} \times \frac{5}{11} = \frac{7 \times 5}{4 \times 11} = \frac{35}{44}$$

Fig. 2.10. Variety of examples showing the procedure for dividing fractions

Students benefit by continuing to use various models for representing situations that involve multiplication and division of fractions even after they begin to use formal symbolic representations. By using pictures, including number lines, to represent the situation, students can avoid the confusion that often occurs when deciding whether to multiply or divide by a fraction to solve a problem.

Decimals

As part of a curriculum that focuses on multiplication and division of decimals, discussions about the equivalence between fractions and decimals, as well as patterns that emerge when decimals are multiplied or divided by powers of 10, are essential.

Reflect As You Read

What opportunities and strategies do you have in your curriculum to discuss equivalence between fractions and decimals?

Fraction and decimal equivalence

A focused discussion of multiplication and division of decimals should begin with an understanding of the basic idea of equivalence between fractions and decimals. (See *Focus in Grade 4*, pp. 41–44.) To expand their understanding of this equivalence relationship, students must also revisit their understanding of division. In grade 6, students learn to express division in different ways, as shown in figure 2.11.

10 divided by 5 can be expressed as $10 \div 5$, $\frac{10}{5}$, and $5\overline{)10}$.

2 divided by 3 can be expressed $2 \div 3$, $\frac{2}{3}$, and $3\overline{)2}$.

a divided by b (for $b \neq 0$) can be expressed as $a \div b$, $\frac{a}{b}$, and $b\overline{)a}$.

Fig. 2.11. Different ways to express division

As students learn to divide in situations where the answer is not a whole number, they often encounter such situations as the following.

$$\begin{array}{r} 3 \\ 5\overline{)17} \\ \underline{15} \\ 2 \end{array}$$

It is reasonable to take this computational work and say, "5 goes into 17 three times with a remainder of 2." But one should not write $17 \div 5 = 3$ r 2, since no meaning for the equality sign makes this statement correct. Instead, the information given by the computation is $17 = 5 \times 3 + 2$. We can divide both sides of the equation by 5 and get

$$17 \div 5 = \frac{17}{5} = \frac{5 \times 3 + 2}{5} = 3 + \frac{2}{5} = 3\frac{2}{5} = 3\frac{4}{10} = 3.4.$$

When the result of whole-number division is not a whole number, the answer can be expressed in three ways, as shown in figure 2.12.

- $17 = 3 \times 5 + 2$ (described as 17 is three 5s with 2 left over)
- $17 \div 5 = 3\frac{2}{5}$ (The quotient is $3\frac{2}{5}$.)
- $17 \div 5 = 3.4$ (The quotient is 3.4.)

Fig. 2.12. Ways to show the result of a whole-number division if the answer is not a whole number

As students learn about different representations of quotients, they realize that when a whole number is divided by a whole number, three results are possible:

- Sometimes the remainder is 0 and sometimes it is not.
- If the answer is not a whole number, then sometimes the quotient is less than 1 and sometimes it is not (i.e., it is a "mixed" number).
- If the answer is not a whole number, sometimes the quotient is a fraction in "lowest terms" and sometimes it is not.

Once students learn that a quotient can be represented by a fraction, they are ready to learn that the converse is also true; a fraction can be interpreted as a quotient, for example,

$$\frac{4}{5} = 4 \div 5.$$

This understanding of the relationship between fractions and quotients enables students to begin to use division as a method to extend their understanding of equivalent fractions and decimals.

Before grade 6, students begin to learn that an equivalence relationship exists between decimals and fractions and that every decimal has a fraction equivalent that can be determined by the place-value positions of the decimal. In grade 6, students apply the connection between rational numbers (fractions with numerators and denominators that are integers; denominator ≠ 0) and their division representations and divide the numerator of a fraction by its denominator to convert fractions to decimals that are either repeating or terminating.

Students in grade 6 are expected to be familiar with some simple fractions and their repeating decimal equivalents, such as

$$\frac{1}{3} = 0.\overline{3}.$$

Students also begin to develop an understanding about why some denominators result in repeating decimals and some do not. For example, through many experiences converting fractions to decimals, they can find patterns that show that fractions that have denominators that can be factored into powers of 2 and/or 5 (such as 2, 4, 5, 8, and 10) result in terminating decimals. Teacher-led discussions can highlight that these denominators have no prime factors other than 2 and 5 and can therefore be multiplied by 2s or 5s to become powers of 10, which then become the denominators for a decimal representation. Examples are shown in figure 2.13.

$$\frac{13}{5} = \frac{13 \times 2}{5 \times 2} = \frac{26}{10} = 2\frac{6}{10} = 2.6 \quad \text{and} \quad \frac{7}{8} = \frac{7 \times 125}{8 \times 125} = \frac{875}{1{,}000} = 0.875$$

Fig. 2.13. Examples of how fractions with denominators that have factors of only 2 or 5 result in terminating decimals

Place-value patterns in multiplying and dividing by powers of 10

Another important skill that helps students in grade 6 understand how to multiply and divide decimals is the ability to apply place-value patterns when multiplying or dividing decimals by powers of 10, as shown in figure 2.14.

- 0.46 × 10 = 4.6
- 0.46 × 100 = 46
- 0.46 × 1,000 = 460

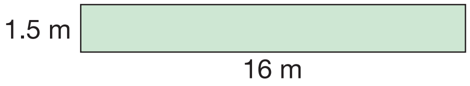
- 4.6 ÷ 10 = 0.46
- 4.6 ÷ 100 = 0.046
- 4.6 ÷ 1,000 = 0.0046

Fig. 2.14. Place-value patterns that result when decimals are multiplied and divided by powers of 10

When multiplying by greater and greater powers of 10, as in the multiplication examples in figure 2.14, the digits in the number move to greater and greater place-value positions. In the division examples, when dividing by greater and greater powers of 10, the digits move to lesser and lesser place-value positions.

Multiplication with decimals

When students first approach multiplying decimals, they commonly draw on their understandings of multiplication of whole numbers and estimation. The following classroom discussion shows how a student may approach a decimal multiplication problem using his or her previous knowledge.



Teacher: Find the area of a rectangular sidewalk that is 1.5 meters wide and 16 meters long.

Carlos: I am not sure what to do with the decimal, but I know that 1.5 × 16 is 1.5 groups of 16. I know that 1 × 16 = 16 and 2 × 16 = 32, so 1.5 × 16 is between 16 and 32. I even know that it is 16 and half of another 16, or 24.

The teacher can then guide Carlos toward a more formal procedure for multiplying decimals by reminding him of other skills that he has learned that he can apply to the problem, as shown in this continued discussion.

Teacher: You did a good job using number sense to estimate a reasonable answer and solve the problem, Carlos. But what if the dimensions were 3.82 meters and 5.1 meters? It might not be as easy to reason out the exact answer. Remember, you know how to use place value to multiply whole numbers, and you have learned a lot about place-value patterns when multiplying or dividing decimals by powers of 10. We can use these ideas to help understand a way to multiply with decimals. How can you use a power of 10 to change 1.5 to 15?

Carlos: I can multiply 1.5 by 10 to get 15.

Teacher: And do you know how to find 15 × 16?

Carlos: Yes, I know a process for that. [Writes for a while.] I get 240.

Teacher: That's right. So to find the product of 1.5 × 16, you can find the product of 15 × 16 = 240, which will be 10 times too big, because it is (1.5 × 10) × 16. So if we divide the product 240 by 10, we get 24.0, or 24, so

$$1.5 \text{ m} \times 16 \text{ m} = 24 \text{ m}^2.$$

It is important in conversations like these to emphasize the impact of the change in the factor on the product. That is, when you multiply or divide a factor by a number, you need to be aware that you have multiplied or divided the product by the same number. The idea of "compensation" can lead to an explanation of the rule in which the numbers of decimal places in the factors are counted and added to find the number of decimal places in the product. Figure 2.15 shows an example of this type of explanation.

Problem: 0.5 × 0.09

Solution: If I think of this as 5 × 9, then I have multiplied the first factor by 10 (one decimal place) and the other factor by 100 (2 decimal places). Since I have increased the product by 10 × 100, or 1,000 (3 decimal places), I have to compensate by dividing the product 45 by 1,000 (3 decimal places) to get 0.045.

Fig. 2.15. Using compensation to multiply decimals

Through a variety of carefully crafted examples that support students as they learn the process, students will begin to see the relationship between the number of decimal places in the factors and the number of decimal places in the product, as shown in figure 2.16.

14 × 16 = 224 16 0 decimal places × 14 + 0 decimal places 224 0 decimal places	1.4 × 16 = 22.4 16 0 decimal places × 1.4 + 1 decimal place 22.4 1 decimal place	0.14 × 16 = 2.24 16 0 decimal places × 0.14 + 2 decimal places 2.24 2 decimal places
1.4 × 1.6 = 2.24 1.6 1 decimal place × 1.4 + 1 decimal place 2.24 2 decimal places	0.14 × 1.6 = 0.224 1.6 1 decimal place × 0.14 + 2 decimal places 0.224 3 decimal places	0.14 × 0.16 = 0.0224 0.16 2 decimal places × 0.14 + 2 decimal places 0.0224 4 decimal places

Fig. 2.16. Examples that show the relationship between the number of decimal places in the factors and the number of decimal places in the product

Although students at this grade level can use estimation and reasonableness to justify the process for finding decimal products, teachers should understand that the equivalence relationship between decimals and fractions is the basis of any efficient, generalizable procedure for multiplying decimals. Figure 2.17 presents specific examples of this relationship as well as generalized, algebraic ones.

$$0.5 \times 0.09 = \frac{5}{10} \times \frac{9}{100} = \frac{45}{1000} = 0.045$$

$$0.08 \times 0.05 = \frac{8}{100} \times \frac{5}{100} = \frac{40}{10{,}000} = 0.0040 = 0.004$$

$$0.008 \times 2.42 = \frac{8}{1000} \times 2\frac{42}{100} = \frac{8}{1000} \times \frac{242}{100} = \frac{1936}{100{,}000} = 0.01936$$

$$0.a \times 0.b = \frac{a}{10} \times \frac{b}{10} = \frac{(a \times b)}{100}$$

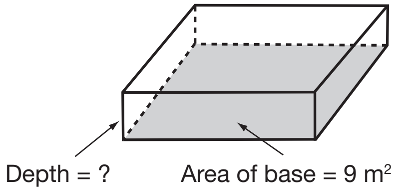
$$0.a \times 0.0b = \frac{a}{10} \times \frac{b}{100} = \frac{(a \times b)}{1000}$$

$$0.0a \times 0.0b = \frac{a}{100} \times \frac{b}{100} = \frac{(a \times b)}{10{,}000}$$

Fig. 2.17. Examples of how the equivalence relationship between fractions and decimals can be used to verify the standard procedure for multiplying decimals

Division with decimals

When students first approach division of decimals, it is common for them to draw on their experiences with multiplication of decimals, as evidenced in this classroom discussion.



Teacher: A playground sandbox is in the shape of a rectangular prism. The area of the base is 9 square meters. If 4.5 cubic meters of sand is dumped into the sandbox and leveled so that it has a uniform depth, what is its depth?

Cindy: I know. This is a volume problem. I remember learning that volume was the number of cubic units that could fit inside a figure. When I stacked cubes in a rectangular prism, I put a layer of cubes on the bottom and the rectangular prism was made up of congruent layers of this first layer. So, if 9 m^2 is the first layer and 4.5 m^3 is the volume of the whole rectangular prism, I need to find out how many 9s are in 4.5. That is division.

Teacher: Interesting reasoning, Cindy. How will you find the quotient?

Cindy: Well, I have to find the number of 9s in 4.5, or 4.5 ÷ 9, and I see that 4.5 is a decimal. I know that when I multiplied by decimals, I thought about how 4.5 is 45 divided by 10. So maybe I can divide 45 by 9 and then divide the quotient by 10. Let me try and see if it makes sense: 45 ÷ 9 = 5 and 5 ÷ 10 = 0.5, so 4.5 ÷ 9 = 0.5.

Teacher: Good, Cindy. Now how can you decide if your answer makes sense?

Cindy: Well, I know that 4.5 is less than 9, so if I divide 4.5 into 9 groups, there must be fewer than 1 in a group. And 0.5 is less than 1, so that is reasonable.

Although Cindy's solution works for this problem, it would not work if she multiplied or divided the divisor by a power of 10. Also, she knows her answer should be less than 1, but that does not give her a good sense about whether the answer of 0.5 is correct, just that it, too, is less than 1. Therefore, teachers need to work with students to help them develop more generalizable approaches when dividing decimals, such as connecting to the relationship between division and fraction representations.

Students have already developed an understanding that a fraction can be interpreted as the answer to a division problem and that a division problem can be written as a fraction. This understanding will be most valuable to students as they learn about dividing decimals. Consider the following classroom discussion.

Teacher: Let's think of another way we can solve the Sandbox Problem. We have to find 4.5 divided by 9. Let's use what we know about division and fractions. Think of the whole-number division problem 45 ÷ 9. How would you write that as a fraction?

Margo: $\frac{45}{9}$.

Teacher: That's right. You can write any division problem as a fraction with the dividend in the numerator and the divisor in the denominator. So we can write 4.5 ÷ 9 as

$$\frac{45}{9}.$$

We can write any type of number as the numerator and denominator of a fraction—whole numbers, decimals, or even other fractions. Once we have written the quotient as a fraction, we can treat the number like any other fraction. For example, we can create an equivalent fraction by multiplying by some fraction form for 1, which is the same as multiplying the numerator and denominator by the same number. Margo, is there a fraction form of 1 that you would multiply 4.5/9 by to make it easier to find the quotient?

Margo: I might multiply by 10/10; that will make 4.5 a whole number, and then the problem is just like a whole-number division problem.

Teacher: That is a good choice, Margo. So multiply by 10/10 to find an equivalent fraction that does not have a decimal in it:

$$\frac{4.5}{9} \times \frac{10}{10} = \frac{4.5 \times 10}{9 \times 10} = \frac{45}{90}.$$

So the quotient of 4.5 ÷ 9 is the same as the quotient of 45 ÷ 90.

Margo: Wait, I see the answer now. I know that 45 is half of 90, so

$$\frac{45}{90} = \frac{1}{2}.$$

I also know that 1/2= 0.5, so 4.5 ÷ 9 = 0.5. That is the same answer that Cindy got!

Once students have been given multiple opportunities to explore dividing a decimal by a whole number by writing the quotient as a fraction, they should be able to use their understanding to divide a decimal by a decimal as well. Consider the example in figure 2.18.

What is 4.5 ÷ 0.9?

Write 4.5 ÷ 0.9 as a fraction: $\frac{4.5}{0.9}$

Multiply numerator and denominator by the same number to get an equivalent fraction with a whole-number divisor:

$$\frac{4.5}{0.9} \times \frac{10}{10} = \frac{4.5 \times 10}{0.9 \times 10} = \frac{45}{9}.$$

So 4.5 ÷ 0.9 has the same quotient as 45 ÷ 9.

45 ÷ 9 = 5, so 4.5 ÷ 0.9 = 5.

Fig. 2.18. Example that shows how writing the quotient of two decimals as a fraction helps students divide

Additionally, through a variety of carefully constructed experiences in which students find quotients in related division problems, they will begin to understand the standard procedures for dividing one finite decimal into another finite decimal, as shown in figure 2.19.

Teachers should encourage students to write the decimal quotients and their fractional representations side-by-side so that they begin to see patterns, as seen in figure 2.20.

As students' understanding of the conceptual basis for the division procedure deepens, they will begin to use zeros to extend the place-value positions in the dividend to continue to divide, when appropriate. As a result, students will encounter the fact that some quotients are terminating decimals, whereas some appear to continue on and on, but do so in a repeating pattern. Extensive experiences with different types of problems will help students determine to what degree of accuracy they need to give the answer; in other words, at what point can they stop dividing and round to give an estimate of the quotient.

Although students at this grade level can use their understanding of writing quotients as fractions to justify the process for finding decimal quotients, When dividing a finite decimal number by another finite decimal number, the following guidelines reduce the work to dividing one whole number into another whole number:

- If the divisor (denominator) is not a whole number, obtain an equivalent problem by multiplying both dividend (numerator) and divisor (denominator) by the same power of 10, chosen so that the new divisor is a whole number. Example:

$$3.55 \div 0.25 = \frac{3.55}{0.25} = \frac{3.55}{0.25} \times \frac{100}{100} = \frac{355}{25} = 355 \div 25.$$

- Divide as you would normally divide if the divisor was a whole number, being careful to use your understanding of place value to align the digits in the quotient with the appropriate place in the dividend.

$$355 \div 25$$

$$\begin{array}{r} 14.2 \\ 25\overline{)355.0} \\ -\underline{25} \\ 105 \\ -\underline{100} \\ 50 \\ \underline{-50} \\ 0 \end{array}$$

- Place the decimal point in the quotient directly above the decimal point in the dividend.

$$\begin{array}{r} 14.2 \\ 25\overline{)355.0} \end{array}$$

Fig. 2.19. Procedure for dividing finite decimals

$$3.6 \div 24 = \frac{3.6}{24} \rightarrow 24\overline{)3.60}$$

$$7 \div 2.8 = \frac{7}{2.8} = \frac{7}{2.8} \times \frac{10}{10} = \frac{7 \times 10}{2.8 \times 10} = \frac{70}{28} \rightarrow 28\overline{)70}$$

$$0.005 \div 0.15 = \frac{0.005}{0.15} = \frac{0.005}{0.15} \times \frac{100}{100} = \frac{0.005 \times 100}{0.15 \times 100} = \frac{0.5}{15} \rightarrow 15\overline{)0.5}$$

Fig. 2.20. Division involving decimals written as fractions and in long-division form

teachers should understand that, as in multiplying decimals, the equivalence relationship between decimals and fractions is the basis of any efficient, generalizable procedure for dividing decimals. Figure 2.21 presents a specific example of this relationship as well as a general, algebraic one.

$$4.5 \div 0.9 = 4\frac{5}{10} \div \frac{9}{10} = \frac{45}{10} \times \frac{10}{9} = \frac{450}{90} = 5$$

For one-digit numbers a and b, $b \neq 0$:

$$0.a \div 0.0b = \frac{a}{10} \div \frac{b}{100} = \frac{a}{10} \times \frac{100}{b} = \frac{a \times 100}{10 \times b}$$

Fig. 2.21. Examples of how the equivalency relationship between fractions and decimals is related to dividing decimals

Strengthening Multiplication and Division of Fractions and Decimals through Problem Solving

Rather than just introduce students to a plethora of unrelated multiplication and division problems with fractions and decimals, the problems that students solve should be focused and purposeful. Clusters of problems should be crafted to help students analyze patterns and internalize relationships. In focused problem-solving opportunities, students are given problems with similar contexts or similar numbers. By analyzing how changes in the problem, context, or numbers affect the solution, students learn to choose the correct operation and develop the ability to determine a reasonable answer to the problem. Students will also gain the confidence to be able to decide how to interpret the answer on the basis of the context. For example, should they write the answer as a decimal, a fraction, or a whole number with a remainder? Is the remainder the answer to the question? Does the remainder affect the answer to the question? Should they disregard the remainder when answering the question? Problem sets like the ones given in figure 2.22 offer students these kinds of opportunities.

By providing related problem clusters like those in set 1, teachers can help students distinguish between contexts and solution approaches in which multiplication is appropriate and ones in which division is appropriate. Students should also be encouraged to think about whether their answers should be greater than or less than the numbers in the problem. For example, in problem 3, students could think that they are dividing 12 by a number less than 1, so they know the answer must be greater than 12. In problem 4, students could think that they are pouring 12 servings, but each serving is less than 1 liter, so the total amount of juice is less than 12 liters.

The problems in set 2 give teachers the opportunity to guide students to look for relationships in the problem by encouraging them to compare the solution with the numbers in the problem and explain why the relationship exists. After many experiences with problem pairs like these, students should be able to explain that when dividing a number (the dividend) by a number less than 1 (the divisor), the quotient is greater than the number that was divided (the dividend). Also, when dividing a number (the dividend) by a number greater than 1 (the divisor), the quotient is less than the number that was

Set 1
To emphasize similar contexts, different operations

1. Five friends share 12 granola bars equally. How many granola bars does each friend get?

Solution: $12 \div 5 = 2\frac{2}{5}$; each friend gets $2\frac{2}{5}$ granola bars.

2. Ms. Ames gives $\frac{2}{3}$ of a granola bar to each of 5 students. How many granola bars does Ms. Ames give out in all?

Solution: $5 \times \frac{2}{3} = 3\frac{1}{3}$; Ms. Ames gives out $3\frac{1}{3}$ granola bars in all.

3. Janie is pouring 0.4-liter servings of juice. She has 12 liters of juice. How many servings can she pour?

Solution: $0.4\overline{)12.0}$ is replaced by $\overset{30}{4\overline{)120}}$. There are 30 servings.

4. Janie poured 12 0.4-liter servings of juice. How much juice did Janie pour?

Solution: $12 \times 0.4 = 4.8$. Janie poured 4.8 liters of juice.

Set 2
To emphasize similar contexts, same operation, and different numbers

1. Jon has $3\frac{1}{2}$ cups of chocolate chips to make cookies. The recipe calls for $\frac{1}{3}$ cup for a single batch. How many batches can Jon make?

Solution: $3\frac{1}{2} \div \frac{1}{3} = 10\frac{1}{2}$; Jon can make $10\frac{1}{2}$ batches.

2. Jon has $\frac{1}{4}$ cup of chocolate chips to make cookies. The recipe calls for $\frac{3}{4}$ cup for a single batch. How many batches can Jon make?

Solution: $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$; Jon can make $\frac{1}{3}$ of a batch.

Fig. 2.22. Examples of sets of related problems

Set 3
To emphasize that you can choose to use decimals or fractions to solve a problem

1. Eric is making $\frac{1}{4}$-pound burgers. He has 1.8 pounds of ground meat. How many whole burgers can Eric make?

Solution: $1\frac{8}{10} \div \frac{1}{4} = \frac{18}{10} \div \frac{1}{4} = \frac{18}{10} \times \frac{4}{1} = \frac{72}{10} = 7\frac{2}{10}$ or

$$1.8 \div 0.25 = \frac{1.8}{0.25} = \frac{1.8}{0.25} \times \frac{100}{100} = \frac{1.8 \times 100}{0.25 \times 100} = \frac{180}{25} = 180 \div 25;\ \overset{7.2}{25\overline{)180.0}};$$

he can make 7 burgers from the package.

2. Eric is making 25 hamburgers for a barbecue. the host of the party told him that he has already made four tenths of the hamburgers. How many burgers has Eric made?

Solution: $\frac{4}{10} \times 25 = \frac{4}{10} \times \frac{25}{1} = \frac{100}{10} = 10$ or $25 \times 0.4 = 10$; Eric has made 10 hamburgers.

Fig. 2.22. Examples of sets of related problems—*Continued*

divided (the dividend). As they progress through their mathematics education, they will discover that these ideas are true only when the numbers are all positive and that this relationship can be represented algebraically as follows: If $a > 0$ and $0 < b < 1$, then $a \div b > a$; and if $a > 0$ and $b > 1$, then $a \div b < a$.

To use the problems in set 3 as a clustered learning experience, teachers can ask students to consider whether they would use decimals or fractions to solve the problems. Students should realize that in the first problem, 1/4 can be written as 0.25 and 1.8 can be written as 1 8/10. In the second problem, students should realize that four tenths can be written as 0.4 or

$$\frac{4}{10} = \frac{2}{5}.$$

Strengthening Understanding through Connections

As students are exposed to connections among mathematical concepts, their conceptual understanding of those concepts deepens. Connections with other mathematical ideas at this grade level are evident throughout this Focal Point.

As students explore division of fractions, they encounter the idea of multiplicative inverses. When students “invert and multiply” to divide fractions, they are actually using the reciprocal, or multiplicative inverse, of the divisor. For example, to find

$$\frac{3}{4} \div \frac{2}{5},$$

students can multiply by the reciprocal of $\frac{2}{5}$:

$$\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2}.$$

The fraction 5/2 is the multiplicative inverse of 2/5 because

$$\frac{2}{5} \times \frac{5}{2} = 1.$$

Another connection inherent throughout this Focal Point is the idea of estimation. Students estimate to check their answers. In long division, they may estimate to determine and place the digits in the quotient. Students can also use their understanding of the equivalence relationship between fractions and decimals to help them estimate and check for reasonableness. The uses of well-developed estimation skills are endless. The chart in figure 2.23 shows some of the many ways that estimation can help students refine their computational skills.

Problem:	***Solution:***
Is 701.12 a reasonable answer for 1.4×50.08?	No; 1.4 is approximately halfway between 1 and 2, $1 \times 50 = 50$, and $2 \times 50 = 100$. Therefore, the product is between 50 and 100.
How can you estimate $9.1 \div 11.9$?	When you round each factor to the nearest whole number, you get $9 \div 12$. Since $9 \div 12 = \frac{9}{12} = \frac{3}{4} = 0.75$, $9.1 \div 11.9$ is about 0.75. So the quotient is close to 0.75.
How can you estimate $5\frac{7}{8} \times \frac{7}{12}$?	When you round each factor to the nearest half, you get $6 \times \frac{1}{2}$. Since $6 \times \frac{1}{2} = 3$, $5\frac{7}{8} \times \frac{7}{12}$ is about 3. So the product is close to 3.
Is 10.1061 a reasonable answer for 37.43×0.27?	Yes. You can think of 0.27 as about 0.25, and $0.25 = \frac{1}{4}$, and 37.43 as 36, so the problem will have about the same answer as $36 \times \frac{1}{4}$. Since multiplying by $\frac{1}{4}$ is the same as dividing by 4, $36 \times \frac{1}{4} = 36 \div 4 = 9$, so 37.43×0.27 is about 9, and 10.1061 is reasonable.

Fig. 2.23. Ways to use estimation when multiplying and dividing fractions and decimals

Connections in later grades

Fluency with operations on whole numbers, decimals, and fractions forms the foundation for much of the future mathematics that students will encounter. In later grades, students will recognize fractions, percents, and certain decimals as ways of representing rational numbers. They will convert among fractions, decimals, and percents and decide which form to use in a given situation. By developing greater fluency with conversion among the forms, students will have a richer repertoire of tools for problem solving (e.g., knowing that they can find 1/4 of a number to find 0.25 times the number, or knowing that they can find 1/4 of a number to estimate 0.27 times the number).

Students will use fractions as they extend their work with ratios to develop an understanding of proportionality. Fluency with equivalent forms of numbers (fractions, decimals, and percents) will help them solve proportions and a variety of percent problems. They will learn that slope, which can be expressed as a fraction or a decimal, is a rate of change. They will add, subtract, multiply, and divide fractions and decimals as they formulate and solve linear equations and use these equations to solve problems. They will extend their knowledge of whole number and decimal place value to include numbers from millionths through millions. They will use scientific notation (which uses decimals and powers of 10) to represent very large and very small numbers.

Developing Depth of Understanding

After reading about focusing on multiplication and division of fractions and decimals as suggested in the grade 6 Number and Operations Focal Point, consider what activities you do or plan to do to discover which models and understandings students already have about multiplication and division of fractions and decimals. How can you move them to more powerful models and deeper understandings before teaching them the formal procedures?

3 Focusing on Ratio and Rate

In grade 6, students develop an understanding of ratio and rate. The focus is on understanding how ratio and rate are connected to multiplication and division. The eventual goal of this Focal Point is for students to be able to use ratio and rate to describe relationships and use equivalent ratios to solve a variety of problems.

Instructional Progression for Ratio and Rate

The focus on ratio and rate in grade 6 is supported by a progression of related mathematical ideas before and after grade 6, as shown in table 3.1. To give perspective to the grade 6 work, we first discuss some of the important ideas that students focused on before grade 6 that prepare them for understanding ratio and rate in grade 6. At the end of the detailed discussion of this grade 6 Focal Point, we present examples of how students will use the ratio and rate understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books from NCTM, for example, *Focus in Grade 3, Focus in Grade 4, Focus in Grade 5, Focus in Grade 7,* and *Focus in Grade 8.*

Table 3.1 on the next page represents an instructional progression for the conceptual understanding of multiplication, division, fractions, ratio and rate, and proportions before grade 6, during grade 6, and after grade 6.

Early Foundations in Ratio and Rate

Before grade 6, students are expected to develop an understanding of several concepts that form the foundation for understanding ratio and rate and the connection of ratio and rate to multiplication and division. These concepts include an understanding of, and fluency with, whole-number multiplication and division, recognizing and generating factors and multiples, and understanding fractions and fraction equivalence.

In grade 4 students are expected to develop an understanding of, and fluency with, multiplication of whole numbers. They work with multiplication as scaling; for example, 3 × 4 can be interpreted as a length of 4 "stretched" to be 3 times as long. In grade 6 they extend this understanding to multiplication with fractions. For example, they interpret

$$3 \times \frac{2}{3}$$

Table 3.1

Grade 6: Focusing on Ratio and Rate—Instructional Progression for Ratio and Rate

Before Grade 6	Grade 6	After Grade 6
Students develop an understanding of, and fluency with, multiplication and division of whole numbers. Students recognize and generate factors and multiples. Students develop an understanding of fractions and fraction equivalence.	Students apply multiplicative reasoning to explain the meanings of ratios and rates (considering rate as a special kind of ratio).* Students recognize and use different ratios to describe different aspects of a given situation (e.g., comparing two parts of a set or comparing a part of a set to the whole set). Students model equivalent ratios and rates in a variety of ways and connect their knowledge of equivalent fractions to equivalent ratios.* Students solve ratio and rate problems using a variety of strategies reflecting their understanding of equivalent fractions and multiplication and division (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12").	Students develop an understanding of proportional relationships. Students graph proportional relationships and recognize the graph as a line through the origin with the constant of proportionality as the slope of the line. Students express proportional relationships as $y = kx$ and distinguish them from other relationships, such as $y = kx + b$. Students use understanding of percent as a ratio to solve problems involving discounts, interest, taxes, tips, and percent of increase or decrease. Students develop an understanding of similarity as a geometric relationship in which relationships of lengths within an object are preserved, and use scale factors to solve problems (e.g., in similar figures, maps, enlargement,…). Students use proportionality to understand π and its use in determining the circumference and area of a circle (introduce formulas).** Students use their knowledge of proportionality to solve a wide range of problems involving ratios and rates. Students understand the slope of a line as a ratio.

*Appears in the Grade 6 Connections to the Focal Points (NCTM 2006).
** Appears in the Grade 7 Connections to the Focal Points (NCTM 2006).

as scaling up, where a quantity with a "size" of 2/3 is multiplied by a factor of 3, and interpret

$$\frac{2}{3} \times 3$$

as scaling down, where a quantity with a "size" of 3 is multiplied by a factor of 2/3.

In grade 4, students use place value, basic facts, and patterns to find products involving powers of 10 (10, 100, 1,000, and so on) and multiples of those powers (such as 10, 20, 30, 100, 200, 300, 1,000, 2,000, 3,000). In grade 5, students build on patterns for multiplying by powers of 10 to apply patterns for dividing multiples of powers of 10, for example, 6 ÷ 3 = 2, 60 ÷ 3 = 20, and 600 ÷ 3 = 200.

In grade 5 students learn about representing a quotient with a fraction. For example, 5 ÷ 2 is equal to 5/2. In other words, division problems can be written as fractions, and the fraction also represents the quotient; it is the new kind of number that is needed to answer the division problem, since the answer is not a whole number.

Before grade 6, students develop understandings about fraction and fraction equivalence. In grade 3, students use fractions to represent part of a whole, part of a set, a point on a number line, or a distance on a number line. They compare and order fractions by using models, benchmark fractions, or common numerators or denominators. They also identify equivalent fractions by using models, including the number line. In grade 4 students use such techniques as modeling to justify methods for generating equivalent fractions, such as multiplying or dividing the numerator and denominator by the same nonzero number. Students' exploration of equivalent fractions in grade 4 builds the foundation for understanding fraction and decimal equivalence.

In grade 5, students continue to develop and use their understanding of equivalent fractions as they add and subtract fractions. In grade 5 students also recognize and generate factors and multiples. They learn, for example, that 10, 20, 30, 40, 50, and so on, are multiples of 10 and that 1, 2, 5, and 10 are factors of 10. They use their understanding of factors and multiples to explore prime and composite numbers, common factors, and common multiples.

The development of the previously mentioned concepts prior to grade 6 builds the foundation that students need to understand ratio and rate, connect ratio and rate to multiplication and division, and use ratio and rate to solve problems.

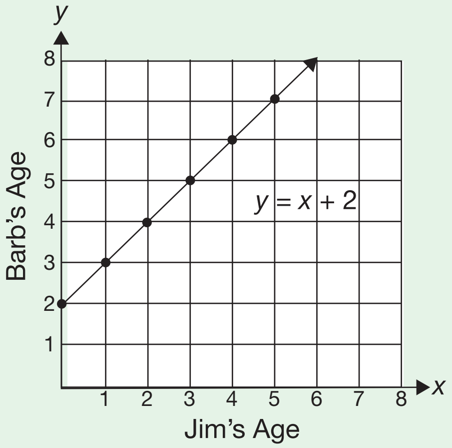
Focusing on Ratio and Rate

To develop students' conceptual understanding of ratio and rate, it is beneficial to start with helping students move from additive to multiplicative reasoning. It is also important to connect ratios to concepts that students already understand, for example, fractions. By linking these two concepts, multiplicative reasoning and fractions, a solid platform for the understanding of ratio and rate can be constructed.

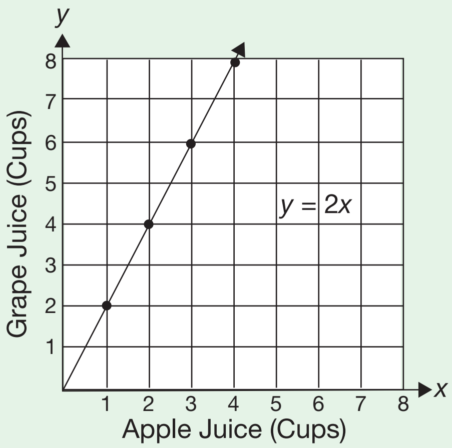
Moving from additive to multiplicative reasoning

Before connecting multiplication and division to the concepts of ratio and rate, students first need to build a connection from additive reasoning to multiplicative reasoning. That is, they need to understand that *2 times* has a different meaning than *2 more than*, *3 times* has a different meaning than *3 more than*, and so on. Students will benefit from constructing various representations of additive relationships and multiplicative relationships as shown in figure 3.1. As students analyze and compare the representations shown on graphs with the same scales, they begin to see the differences between an additive relationship and a multiplicative relationship.

<table>
<tr>
<td><i>Situation:</i>
Barb was 2 years old when her brother Jim was born.</td>
<td><i>Situation:</i>
You use 2 cups of grape juice for every cup of apple juice when you make fruit punch.</td>
</tr>
<tr>
<td><i>Table representation:</i>
<table>
<tr><td>Jim’s age (<i>x</i>)</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>Barb’s age (<i>y</i>)</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
</table>
</td>
<td><i>Table representation:</i>
<table>
<tr><td>Cups of apple juice (<i>x</i>)</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>Cups of grape juice (<i>y</i>)</td><td>2</td><td>4</td><td>6</td><td>8</td></tr>
</table>
</td>
</tr>
<tr>
<td><i>Graphical representation:</i>



</td>
<td><i>Graphical representation:</i>



</td>
</tr>
<tr>
<td><i>Algebraic representation:</i>
$y = x + 2$</td>
<td><i>Algebraic representation:</i>
$y = 2x$</td>
</tr>
<tr>
<td>An additive relationship</td>
<td>A multiplicative relationship</td>
</tr>
</table>

Fig. 3.1. Example showing an additive and a multiplicative relationship

A useful activity to help students build a connection from multiplication to ratio and rate is to explore multiplication as scaling. When a quantity is multiplied by a number greater than 1, it is scaled up. When a number is multiplied by a positive number less than 1, it is scaled down. (When a number is multiplied by 1, the identity element for multiplication, the number remains the same.) The number by which one multiplies is called the *scaling factor* (or *scale factor*). Opportunities to examine such problems as those shown in figure 3.2 can help students understand the concept of scaling.

Students also need to build a connection from division to ratio and rate. This can be accomplished when they learn that a ratio is a comparison of two

Problem: A cake recipe calls for $\frac{2}{3}$ cup of sugar. How much sugar is needed for 4 cakes? Do you scale up or scale down to solve the problem? What is the scaling factor? ***Solution:*** $4 \times \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}$ cups of sugar are needed for 4 cakes. You are scaling up because you are multiplying by a number greater than 1. The scaling factor is 4.	***Problem:*** A recipe for a gallon of punch calls for $\frac{2}{3}$ cup of lime juice. How much lime juice is needed for $\frac{1}{4}$ gallon of punch? Do you scale up or scale down to solve the problem? What is the scaling factor? ***Solution:*** $\frac{1}{4} \times \frac{2}{3} = \frac{2}{12} = \frac{1}{6}$ cup of lime juice is needed for $\frac{1}{4}$ gallon of punch. You are scaling down because you are multiplying by a number less than 1. The scaling factor is $\frac{1}{4}$.
Scaling up	**Scaling down**

Fig. 3.2. Examples showing scaling up and scaling down with multiplication

numbers by division; for example, the ratio 4:5 can be written 4/5 and is the quotient of 4 ÷ 5. Students will use this relationship when they write ratios as decimals.

Using fractions to build an understanding of ratios

A ratio is a multiplicative comparison of two numbers. As students begin to use what they know about fractions to understand ratios, they will learn, for example, if there are 4 children and 2 adults in a family, the ratio that compares the number of children to adults is 4 to 2. Students will also learn that they can make other comparisons. They can compare the number of children to the total number of people in the family with the ratio 4 to 6, they can compare the number of adults to the total number of people in the family with the ratio 2 to 6, and so on. In general, students will learn that they can write part-to-part ratios, part-to-whole ratios, and whole-to-part ratios. Students will also apply the idea of ratio when working with rates. Although there is no universally accepted definition of *rate,* when discussing the ratio between two measurements, especially measurements involving different units, we often use the word *rate* instead of ratio. Rates are usually expressed using the word *per,* for example, miles per gallon. In later grades, students will also encounter an *extended ratio*, the form of *a*:*b*:*c*, as a way of comparing three or more quantities. For example, the extended ratio 1:2:3 describes the ratio of sand to cement to gravel for a concrete mix. The ratio of the measures of the angles of a triangle might be compared with the extended ratio 1:1:2. Although extended ratios are still ratios, they cannot be written in fraction form.

As students begin their investigation of ratio, it is beneficial to start with a context that gives students a reason to learn about ratio, for example:

> *Problem:* You are planning a menu for a class party. Each student will receive one drink at the party. Your teacher tells you that, for a class this size, you can expect 3 out of 5 students to prefer cola, whereas 2 out of 5 will prefer lemonade. If, in fact, this guideline proves true for your class of 30 students, how many students will want cola? How many will want lemonade?

As the following classroom discussion shows, before students learn about ratios, they may use their prior knowledge of multiplication, division, or fractions to solve this problem.

Reflect As You Read

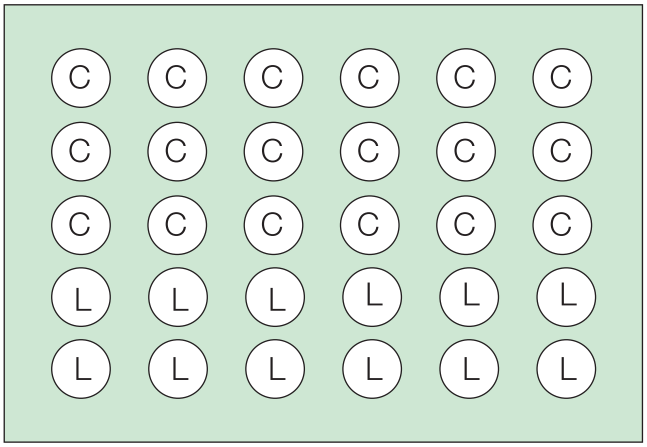
Before continuing, think about the following questions:

How would your students think about this problem?

Can you and your students represent a solution in more than one way? Are some solutions more powerful than others?

Teacher: How did you find the number of colas and the number of lemonades you should order?

Barb: I drew a picture to represent the problem. I drew a column of 5 circles to represent 5 students. I labeled 3 with C for cola and 2 with L for lemonade. Then I drew more of these columns of 5 circles until I had 30 circles.



Teacher: How does your model show the answer?

Barb: My circles show the number of Cs (colas) and Ls (lemonades) for 30 students. There are 18 Cs and 12 Ls, so I would order 18 colas and 12 lemonades.

Teacher: That makes sense. I can see that if you have 18 colas for 30 students and 12 lemonades for 30 students, then you have 3 colas for every 5 students and 2 lemonades for every 5 students. Did anyone else use a drawing to solve the problem?

Stephen: I started to, but I realized that I didn't need to draw all the circles.

Teacher: Tell us why, Stephen.

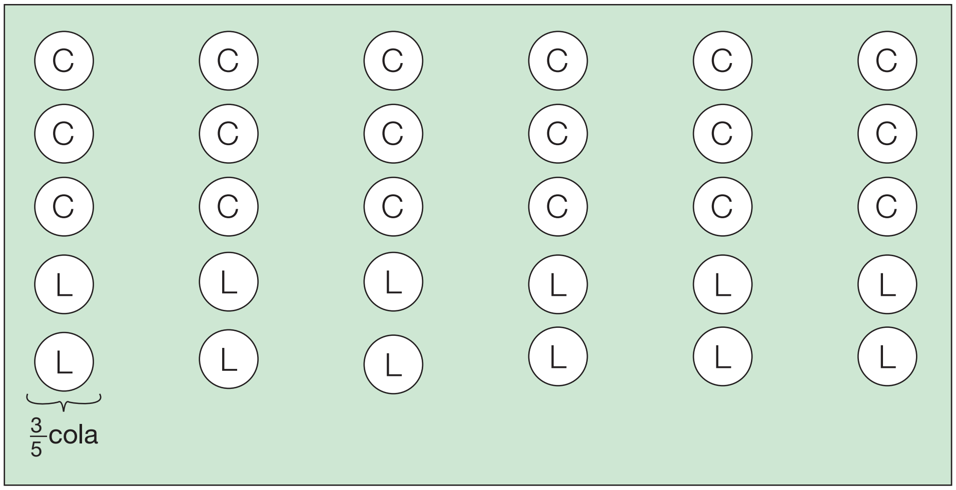
Stephen: I knew I needed 30 circles in all—that's 6 groups because there are 5 in each group. There are 3 Cs in each group, so there are 6 groups of 3 Cs, or 6 × 3 =18 Cs in all. There are 2 Ls in each group, so there are 6 groups of 2 Ls, or 6 × 2 = 12 Ls in all. I used multiplication.

Teacher: I understand. You used division and then multiplication to make sure that there were 3 colas for every 5 students and 2 lemonades for every 5 students. You ended up with 18 colas for 30 students and 12 lemonades for 30 students.

Yonnie: I used division and multiplication, but I didn't draw a model. I know that 30 ÷ 5 = 6, so there are six 5s in 30. That means that I would need six 3s for the number of colas and six 2s for the number of lemonades. Then 6 × 3 = 18, so I need 18 colas, and 6 × 2 = 12, so I need 12 lemonades. I added 18 + 12 = 30, so I know my answer is right.

Teacher: So you also ended up with 18 colas for 30 students and 12 lemonades for 30 students, which is also 3 colas for every 5 students and 2 lemonades for every 5 students.

Trish: I drew the same picture as Barb, but I used fractions to solve the problem.



The fraction of 5 students that want cola is 3/5. The fraction of 5 students that want lemonade is 2/5. Then I used equivalent fractions to find the fraction of 30 students that want cola and lemonade:

$$\frac{3}{5} = \frac{?}{30}$$
$$\frac{3 \times 6}{5 \times 6} = \frac{18}{30}$$

So I need to order 18 colas.

$$\frac{2}{5} = \frac{?}{30}$$
$$\frac{2 \times 6}{5 \times 6} = \frac{12}{30}$$

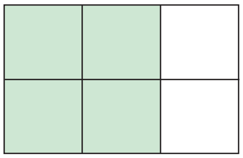
So I need to order 12 lemonades. And 18 + 12 = 30, so my answer is right.

Teacher: Again, there are 18 colas for 30 students (or 3 colas for every 5 students) and 12 lemonades for 30 students (or 2 lemonades for every 5 students). So what we've done today is use the idea called *ratio*. When we describe this situation by saying things like *3 for every 5* and *18 for every 30*, we are using ratio language. We were able to use what we know about fractions and multiplication and division in this ratio situation because those ideas are also related to the idea of comparing some number to another number.

As students are presented with such contexts as the Class Party Problem, teachers can connect students' prior knowledge to representations that support their use of ratios and equivalent ratios to solve the problems. Through experiencing such problems, students will have a context through which to learn and understand ratios.

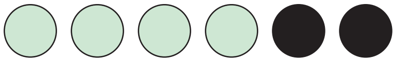
Students in grade 6 understand that a fraction is a way of comparing a part to a whole. They can use this knowledge to develop an understanding of ratios. The link between students' prior understanding of using a fraction to model a part of a whole and the representation of using a ratio to model other relationships, such as comparing a part to a part, can be made through classroom discussions such as the one that follows.

Teacher: Can you make a model to show 4/6 and explain what it means?
Amanda: I used a grid model.



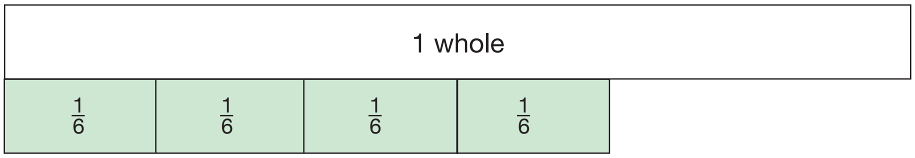
I divided a rectangle into 6 equal parts and shaded 4 of the parts, so 4/6 of the rectangle is shaded.

Brent: I used counters.



My model has 4 green counters and 6 counters in all, so 4/6 of the counters are green.

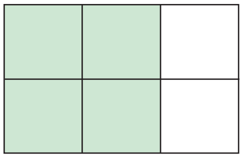
Camila: I used a fraction strip model.



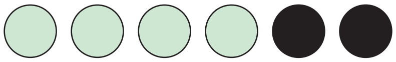
I put 4 sixth strips next to each other and under a one whole strip to show 4/6.

Teacher: All of your models show 4/6. Each model shows 4 equal parts compared to a whole that is the same size as 6 of those equal parts. So each model shows a fraction comparing a part to a whole. A ratio also is a comparison of two quantities, so each of your examples also shows a ratio that compares a part to a whole.

Look at Amanda's model. Her grid model shows that the comparison, or ratio, of shaded squares to all squares is 4 to 6, 4 shaded parts for every 6 parts.



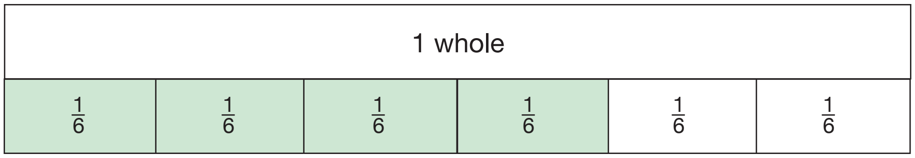
Brent's set model shows that the comparison, or ratio, of green counters to all counters is 4 to 6, 4 green counters for every 6 counters.



To use Camila's model to show a comparison between the number of parts taken (4) and the number of parts in the whole, we need to modify it a little. Camila, what is missing in your diagram that we need to show in the comparison?

Camila: My model only shows the 4 sixths; it doesn't show the other 2 sixths.

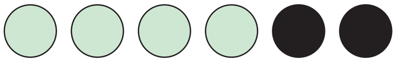
Teacher: That's right. To change the fraction strip model to show a comparison of parts taken to the whole, we need to show all 6 of the sixths in the whole. [Note that the teacher shifts the language here from 4 and 2, which is additive thinking, to 4 and 6, which is the multiplicative relationship that is the focus.] We can circle or shade 4 of the parts to show that 4 of the 6 parts are taken, so you also have 4 shaded pieces for every 6 pieces.



So a ratio can represent a comparison of a part to a whole and can be represented by a fraction.

Discussions such as this one help students link their prior knowledge of a fraction as part *of* a whole to a ratio as a comparison of part *to* a whole. Teachers can build off of this prior knowledge to discuss how ratios can also be used to represent part-to-part and whole-to-part comparisons, as is evident in the continuation of this classroom discussion.

Teacher: A ratio can represent a comparison of a part to the whole, but it can also represent a comparison of a part to another part or of the whole to a part. Let's look at Brent's set model.



We used the ratio 4/6 (4 to 6) to compare the number of green counters to the number of counters in all. But we can make many other comparisons with these counters. Can anyone think of other comparisons we can make?

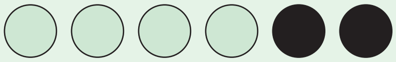
Joshua: We can compare the number of black counters to the number of counters in all. That would be 2/6 (2 to 6).

Teacher: Okay. How could we describe a comparison between the different colors?

Jeff: We can compare green to black. But would you write it 4/2?

Teacher: Yes, we can use the fraction 4/2 to describe the part-to-part ratio, and we think *4 greens for every 2 blacks.* The fraction in this situation does not describe the comparison of a part to a whole; it describes a comparison of a part to a part.

Then the teacher would guide students to write in fraction form the different ratios for each comparison. The comparisons and ratios are summarized in figure 4.3.



Comparison	Ratio
Green to all	$\frac{4}{6}$
Black to all	$\frac{2}{6}$
Black to green	$\frac{2}{4}$
Green to black	$\frac{4}{2}$
All to green	$\frac{6}{4}$
All to black	$\frac{6}{2}$

Fig. 3.3. Ratios that describe four green counters and two black counters

Students should also be exposed to sets that can be separated into more than two parts, for example, the set in figure 3.4.

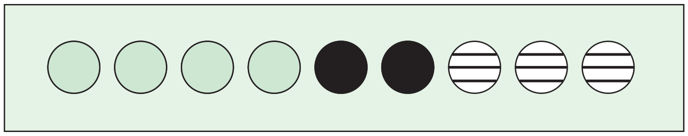


Fig. 3.4. Set that can be separated into 3 different parts

Students learn that even more ratios can be written to compare the elements in this set. For example, students can compare green circles to black circles (4/2), black circles to striped circles (2/3), striped circles to green circles (3/4), all circles to green circles (9/4), black circles to all circles (2/9), and so on.

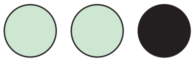
In these contexts, students can learn how to read and write the different formal representations of a ratio. For example, in the counter model comparing 4 green counters with 6 counters in all, the ratio can be written as 4 to 6, 4/6, or 4:6. As students develop their understanding of the ratios used to describe different situations, they can then begin to create situations to represent given ratios. For example, teachers could give students a ratio such as 2:3 and ask them to use counters to create a situation that can be described by the ratio 2:3. As is clear from the following classroom discussion, students will quickly begin to develop an appreciation of the fact that one ratio (e.g., 2:3) can be represented in a variety of ways.

Teacher: Use black and green counters to model the ratio 2 to 3.

Corey: I used 5 counters. My model shows 2 green and 3 black counters: 2 green to 3 black.

(green to black)

Rae: I modeled the same ratio, and I only used 3 counters. Look, I did 2 green counters and 1 black counter; my model shows 2 green to 3 counters in all.



(green to all)

Teacher: Both of your models are correct ways to show the ratio 2:3. Corey's model shows it as a part-to-part ratio. Rae's model shows it as a part-to-whole ratio. Does anyone else have a different model?

Through classroom discussion, students would begin to realize that all the models shown in figure 3.5 are correct ways of modeling the ratio 2:3.

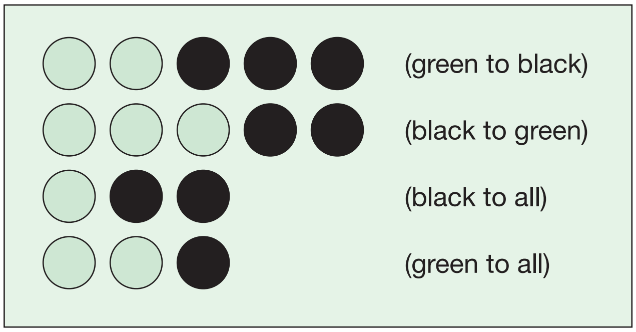


Fig. 3.5. Different ways to represent the ratio 2:3

Experiences such as this one, including ones in which the set contains more than two different elements, will help students realize that the order of the numbers in a ratio is extremely important and that the ratio one wants to use depends heavily on the context.

Reflect As You Read

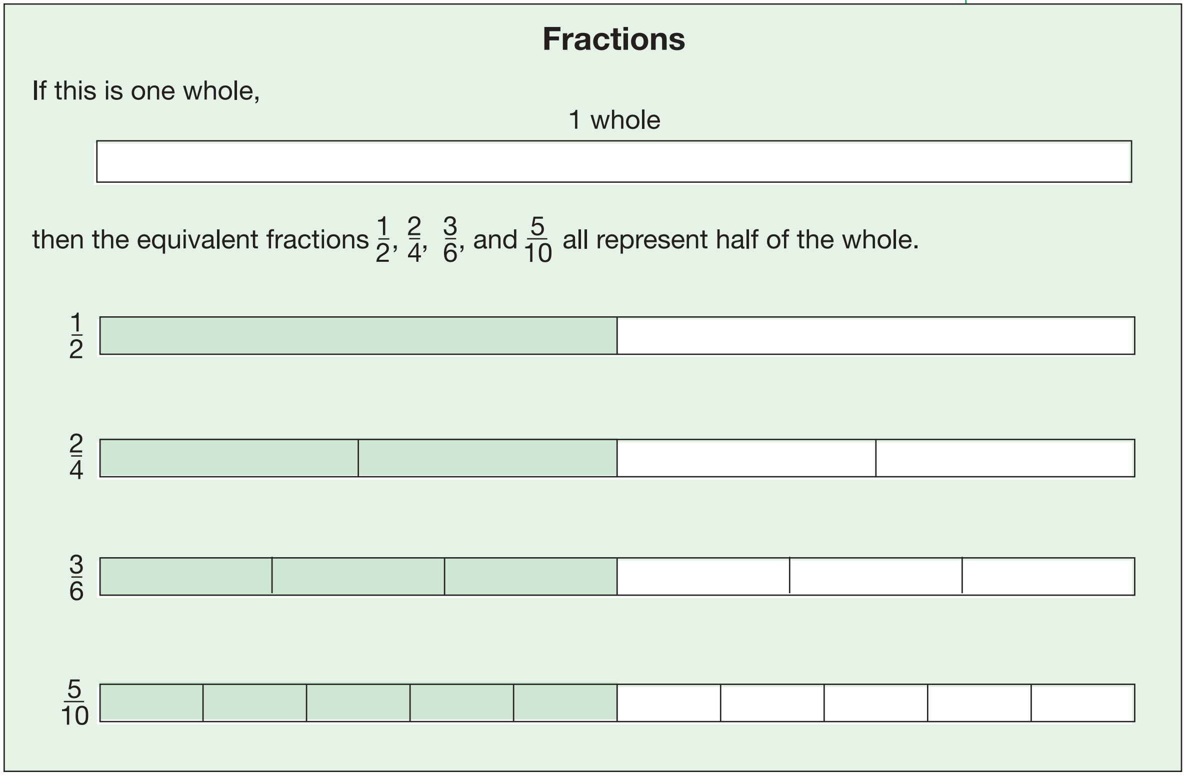
After reading this section about ratios and their meaning, think about these questions:

How does your curriculum material develop these ideas?

Do students have enough time and various experiences to fully understand the ideas of ratio before moving on to related concepts?

Using equivalent fractions to build an understanding of equivalent ratios

Students can use the understanding that they have of equivalent fractions as a basis for their understanding of equivalent ratios. However, teachers must help students make the transition from only thinking about a fraction as representing the comparison of a part to a whole to thinking about a fraction as a representation of many types of ratios. Expanding this thinking is particularly important in dealing with the difference between the representations of equivalent fractions and equivalent ratios. In common representations of equivalent fractions, the whole stays the same. To create the equivalence, the whole is divided into smaller and smaller parts (see fig. 3.6). By contrast, in common representations of equivalent ratios, the whole is replicated over and over, or multiplied (see fig. 3.7).



Fractions

If this is one whole,

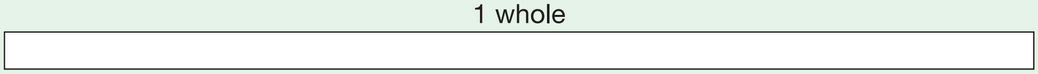
then the equivalent fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{5}{10}$ all represent half of the whole.

Fig. 3.6. Common representations of equivalent fractions

In the equivalent ratios 1:2 = 2:4 = 3:6 = 5:10, the 1-for-every-2 relationship is replicated over and over, or multiplied, and the relationship of the *new* parts to the *new* whole does not change. This concept of multiplication as scaling lays the foundation for proportionality in later grades. Both a fractional amount and a ratio can be represented by the same symbol, for example, 5/10, but their meanings in context can be different.

Ratio

If this is one whole,



the equivalent ratios $\frac{1}{2}$ = 1:2, $\frac{2}{4}$ = 2:4, $\frac{3}{6}$ = 3:6, and $\frac{5}{10}$ = 5:10 can be represented by the following diagrams.

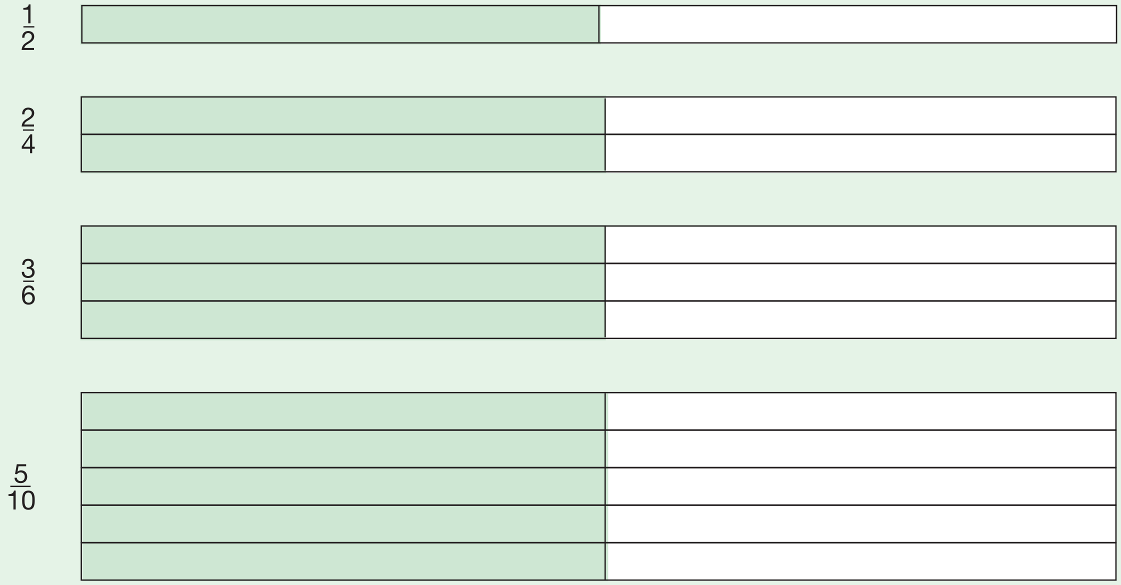
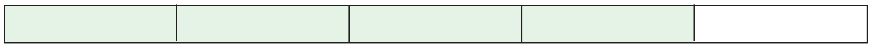


Fig. 3.7. Common representations of equivalent ratios

As previously stated, though, students can use their understanding of fractions to represent equivalent ratios, as can be seen in the following classroom discussion.

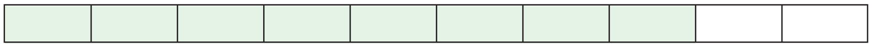
Teacher: Amanda, make a grid model like the one you used to show 4/6 to show 4/5.

Amanda:

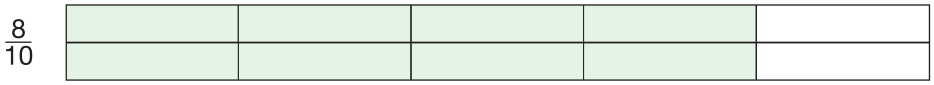


Teacher: Amanda, show me how you would change your model to make it show an equivalent fraction.

Amanda: I could divide each part in half. Then I would have 10 parts with 8 shaded. My model would show the fraction 8/10.

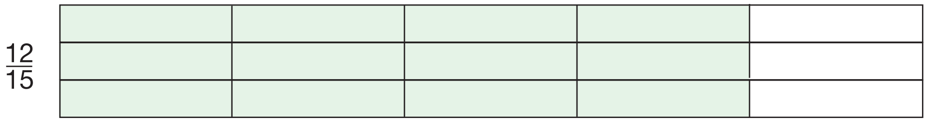


Teacher: That is correct. Your first model not only shows the fraction 4/5, but it shows the ratio 4 to 5, or 4 shaded parts for every 5 parts. The model of the equivalent fraction shows 8/10. You could also read this as a ratio, 8 shaded parts for every 10 parts. But you can also create a model for the ratio 8 to 10 by doubling your 4-out-of-5 model.



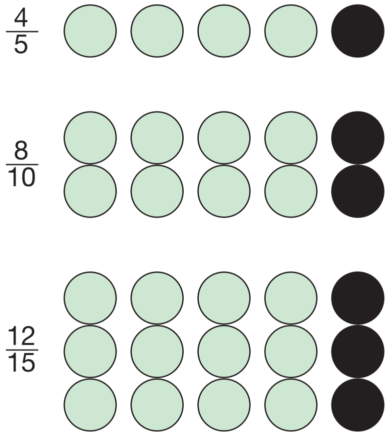
Your new model shows 8 out of 10 squares shaded, or the ratio 8 to 10, and could be described with the fraction 8/10. It still also represents the ratio 4 to 5 and could be described with the fraction 4/5. Can you use your 4/5 model to show the equivalent ratio 12 to 15?

Amanda: Yes, I can. I can copy my model another time to show 12 out of 15 squares shaded.



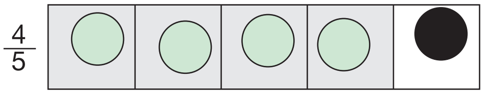
Teacher: And you can label it with the fraction 12/15. That is an interesting picture. You copied your 4 out of 5 model 3 times, and I can quickly see that 12 shaded parts for every 15 parts is equivalent to 4 shaded parts for every 5 parts. So the ratio 12 to 15 is equivalent to the ratio 4 to 5. I think maybe we could copy the 4-for-every-5 model any number of times. Now let's look at Brent's counter model. Brent, can you use your counter model to first show the ratio 4 to 5? And can you repeat the 4-to-5 model to show 8 to 10 and 12 to 15?

Brent: Yes.

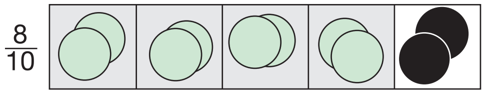


I repeated 4 to 5 to get 8 to 10. Then I repeated 4 to 5 again to get 12 to 15.

Teacher: Now let's combine Amanda's and Brent's models. Place the counters on the grid squares to show the ratio 4 for every 5.

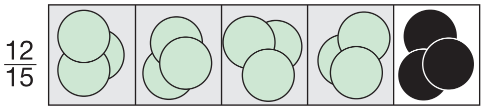


Next, rather than make a new row of grid squares, place another set of counters on top of the ones already in the model. What does the model show now?



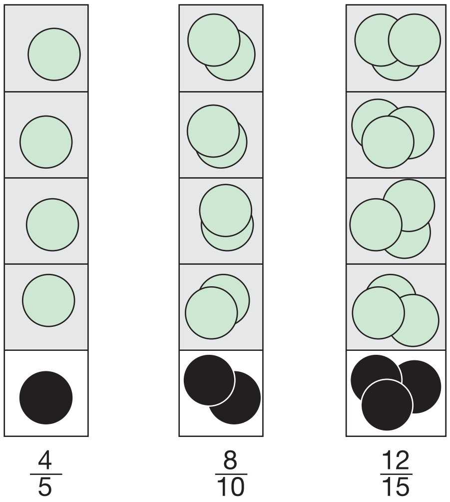
Camila: The model shows 8 to 10 because there are 8 greens and 10 counters.

Teacher: Now use the same method to show the ratio 12 to 15.

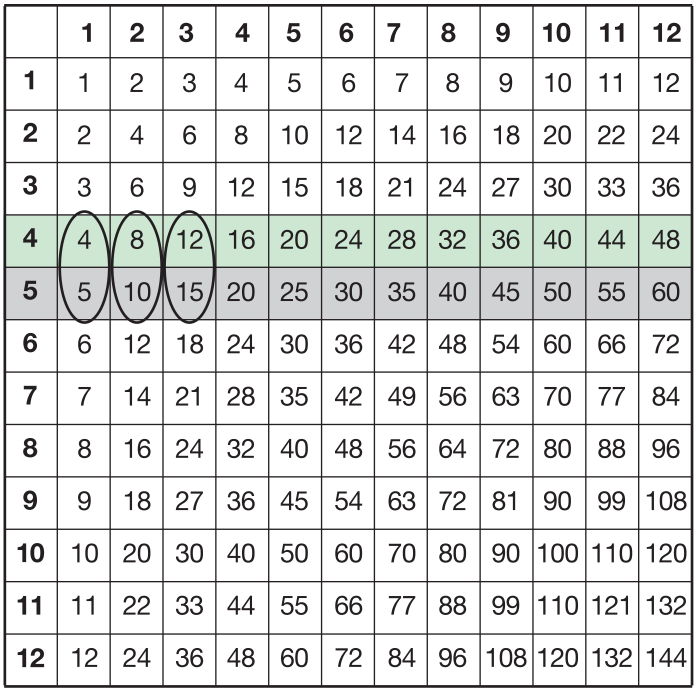


Teacher: Now turn your grid paper so the models appear as columns rather than rows. The models still represent the equivalent ratios 4:5 = 8:10 = 12:15, or in fraction form,

$$\frac{4}{5}=\frac{8}{10}=\frac{12}{15}.$$



Now the teacher can connect this grid model to another powerful model to show equivalent ratios, the multiplication table. At this point in the discussion, the teacher would show a multiplication table, highlight rows 4 and 5, and circle the numbers as shown.



	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Teacher: The rows and columns of the multiplication table show equivalent fractions and, therefore, equivalent ratios. If you follow the rows for 4 and 5 to the right, you will see the same equivalent ratios you modeled: 4/5, 8/10, 12/15, as well as other equivalent ratios:

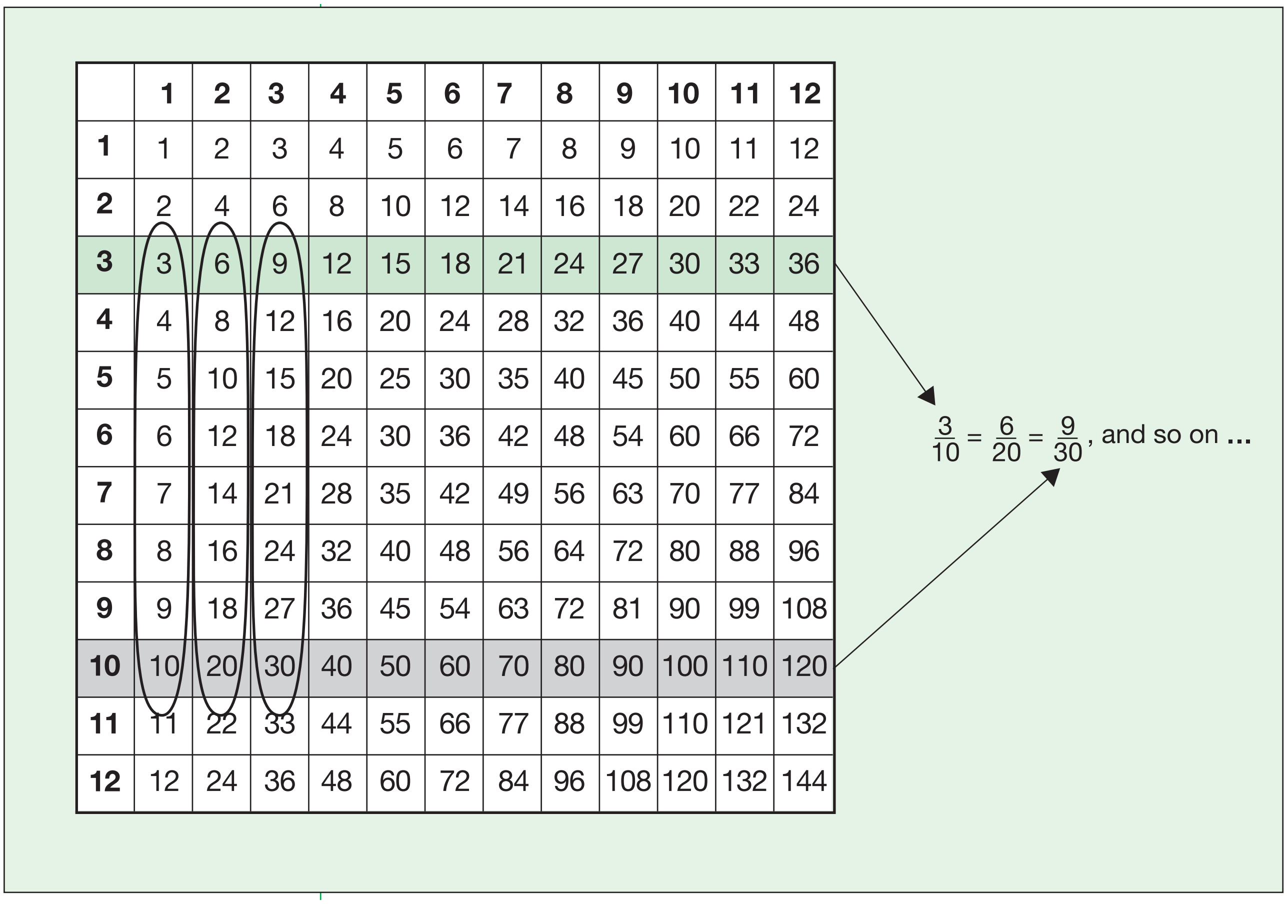
$$\frac{16}{20}=\frac{20}{25}=\frac{24}{30},$$

and so on.

Teachers should point out that any rows or columns in the multiplication table can be paired to show equivalent fractions, even if the rows or columns are not touching, as shown in figure 3.8.

To help students see the equivalent ratios in nonconsecutive rows or columns, such as 8/3, 16/6, and 24/9, teachers can have students cut apart the rows of the table and reposition the rows, one directly above the other.

To gain a thorough understanding of equivalent ratios, students need to have many opportunities to work with and use a variety of models to show equivalence with different kinds of ratios, including part-to-part, part-to-whole, and whole-to-part.



	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

$\frac{3}{10} = \frac{6}{20} = \frac{9}{30}$, and so on ...

Fig. 3.8. How the multiplication table can be used to show equivalent ratios

Reflect As You Read

After reading this section about equivalent fractions and equivalent ratios, think about these questions:

What new ideas about ratio emerged in this section?

How does this discussion help distinguish the important differences between fractions and ratio?

Understanding rate as a special kind of ratio

As students progress through their focused exploration of ratios, they will encounter a special kind of ratio called a *rate.* Although there is no universally accepted definition of *rate,* when discussing the ratio between two measurements, especially measurements involving different units, we often use the word *rate* instead of ratio. This is especially true when one of the measurements involves time. It is also fairly standard practice to describe a rate in terms of the value of one measurement relative to one unit (i.e., whole) of the other unit. Examples of ratios that we would call rates include 55 miles per hour, 3 gallons per minute, $3.47 per pound, $2.59 per gallon, or 300 bushels per acre. In some instances, the denominator may not be just one unit, such as $5 per 10-minute call. In all of these examples, note the use of the word *per*. Here the word is communicating the need for division. Other examples of rates are shown in figure 3.9.

$$\text{Rates:}\quad \frac{80\text{ words}}{2\text{ minutes}}\qquad \frac{48\text{ inches}}{4\text{ feet}}\qquad \frac{100\text{ students}}{4\text{ buses}}\qquad \frac{20\text{ miles}}{3\text{ hours}}$$

Fig. 3.9. Examples of rates

Referring to figure 3.9, in the rate

$$\frac{80\,\text{words}}{2\,\text{minutes}},$$

the amount of work someone did is being described in two ways—as measured by words and as measured by minutes, as seen in the diagram in figure 3.10.

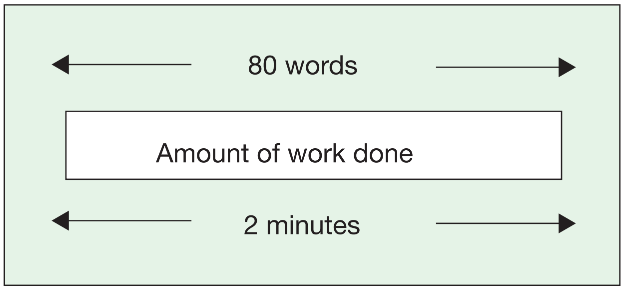


Fig. 3.10. Example of a rate in which a person typed 80 words in two minutes

Students learn that a *unit rate* is a rate with a denominator of 1 unit. Prior to students' learning about ratios and rates, they have used unit rates to solve problems. For example, in figure 3.11, although they did not realize it, when the students divided to find the cost of 1 pound of apples, they were finding the unit rate. Then when they multiplied $2 per pound by 5 pounds to find the solution, they were using the unit rate to solve the problem.

Any rate can be written as a unit rate. One way to write a rate as a unit rate is to perform the division indicated by the fraction, in other words, divide the numerator by the denominator. The quotient becomes the new numerator and the denominator becomes 1. The rates in figure 3.9 are shown in figure 3.12 as unit rates. When writing a rate as a unit rate, one is writing a ratio as an equivalent ratio with a denominator of 1, so in each case, each original rate is equivalent to its unit rate.

Problem:

If 3 pounds of apples costs \$6, how much do 5 pounds of apples cost?

Solution:

Divide 6 by 3 to get the cost of 1 pound of apples.
Then multiply that cost by 5.
$6 \div 3 = 2$, so 1 pound of apples costs \$2.
$2 \times 5 = 10$, so 5 pounds of apples cost \$10.

Fig. 3.11. Example of how students use their prior understanding of unit rate to solve problems

Unit rates: $\frac{\text{40 words}}{\text{1 minute}} \ldots \frac{\text{12 inches}}{\text{1 foot}} \ldots \frac{\text{25 students}}{\text{1 bus}} \ldots \frac{6.\overline{6}\text{ miles}}{\text{1 hour}}$, or $\frac{6\frac{2}{3}\text{ miles}}{\text{1 hour}}$

Fig. 3.12. Examples of unit rates

Unit rates are typically written and read using the word *per*. The unit rates in figure 3.12 would most likely be written and read as 40 words per minute, 12 inches per foot, 25 students per bus, and 6 2/3 miles per hour. Note that the words used to describe unit rates, and the fact that students often see unit rates described with only one number visible, the numerator of the ratio (e.g., 40 words per minute), make it difficult for students to think about unit rates in fraction form.

It is beneficial for students to understand that every rate situation can be written in two ways with two different unit rates, with either unit as 1. For example, the situation in which 6 pounds of bananas costs \$3 can be written as the rate

$$\frac{6 \text{ pounds}}{3 \text{ dollars}}$$

and as the unit rate in terms of pounds per dollar:

$$\frac{2 \text{ pounds}}{1 \text{ dollar}}.$$

The same situation can be described by the rate

$$\frac{3 \text{ dollars}}{6 \text{ pounds}}$$

and as the unit rate in terms of dollars per pound:

$$\frac{0.5\ \text{dollars}}{1\ \text{pound}} \text{ or } \frac{\$0.50}{1\ \text{pound}}.$$

When the unit rate is written in terms of the amount of money for 1 quantity, it is called the *unit cost* or *unit price*. In either instance, to find the unit rate, one must find the equivalent ratio whose denominator is 1 unit. One way to find this equivalent ratio is to divide the numerator by the denominator; another is to use equivalent ratios. In the examples shown in figure 3.13, each situation is described using two rates, and two mathematically equivalent methods of finding each unit rate are shown.

Situation	Rate	Unit Rate (Using Equivalent Ratios)	Unit Rate (Using Division)
The cost of gasoline is \$8 for 2 gallons.	How much does 1 gallon cost? $\frac{8\ \text{dollars}}{2\ \text{gallons}}$ How much can you buy for \$1? $\frac{2\ \text{gallons}}{8\ \text{dollars}}$	$\frac{8}{2}=\frac{8\div 2}{2\div 2}=\frac{4}{1}=4$ \$4.00 per gallon $\frac{2}{8}=\frac{2\div 2}{8\div 2}=\frac{1}{4}$ $\frac{1}{4}$ gallon per dollar	$\begin{array}{r} 4.00 \\ 2\overline{)8.00} \end{array}$ \$4.00 per gallon $\begin{array}{r} 0.25 \\ 8\overline{)2.00} \end{array}$ 0.25 per gallon per dollar
A student runs 300 feet in 20 seconds.	How far in 1 second? $\frac{300\ \text{feet}}{20\ \text{seconds}}$ How many seconds to go 1 foot? $\frac{20\ \text{seconds}}{300\ \text{feet}}$	$\frac{300}{20}=\frac{300\div 20}{20\div 20}=\frac{15}{1}=15$ 15 feet per second $\frac{20}{300}=\frac{20\div 20}{300\div 20}=\frac{1}{15}$ $\frac{1}{15}$ second per foot	$\begin{array}{r} 15 \\ 20\overline{)300} \end{array}$ 15 feet per second $\begin{array}{r} 0.0\overline{6} \\ 300\overline{)20.00} \end{array}$ approximately 0.07 seconds per foot

Fig. 3.13. Examples of finding equivalent unit rates

Reflect As You Read

Before continuing, think of other situations that lead to unit rates. Keep a record of them for work in the classroom.

Strengthening Connections among Multiplication, Division, Fractions, and Ratios through Problem Solving

Students who have had many opportunities to work with the concepts and representations of ratio and rate will be better prepared to use ratio and rate to solve problems. To help students focus on using ratio to solve a problem, it is beneficial to revisit problems involving comparisons that students have previously solved more intuitively and model how to solve the problems using ratios. For example, teachers can revisit the Class Party problem:

> *Problem:* You are planning a menu for a class party. Each student will receive one drink at the party. Your teacher tells you that, for a class this size, you can expect 3 out of 5 students to prefer cola, whereas 2 out of 5 will prefer lemonade. If, in fact, this guideline proves true for your class of 30 students, how many students will want cola? How many will want lemonade?

Recall that before students understood ratios, they used their prior knowledge of representations involving multiplication, division, and fractions to solve this type of problem. Teachers can guide students to connect these ideas to representations of ratios to solve this and other problems, as is shown in the following classroom discussion.

Teacher: Remember how we used drawings and multiplication and division to solve the Class Party problem? [Teacher would review the drawings and procedures.] Now we know that we can use the ideas of ratios and equivalent ratios to solve the problem.

The teacher should then guide students to see how they can use equivalent ratios to find the solution, as shown in fig. 3.14.

Note that in the problem in figure 3.14, students may understand that rather than use equivalent ratios to find both the number of colas and the number of lemonades needed for 30 students, they could find the equivalent ratio for one drink, and subtract that from the total, 30, to find the number of the other drink. For example, if students found that they needed 18 colas for 30 students, then they can surmise that they need 30 – 18, or 12, lemonades. Completing the problem both ways is valuable so that students realize that the sum of the colas and the lemonades equals 30, that is, 18 + 12 = 30.

Students will benefit by solving many problems that facilitate the type of reasoning and discussion shown in the Class Party problem. The discussions should guide students to build on what they know about multiplication, division, and fractions to strengthen their understanding of ratios and equivalent ratios. A variety of models should be encouraged. Students should come to understand how to recognize comparison relationships whose solutions involve ratios and that, given any ratio, a set of equivalent ratios can be formed using multiples of the numerator and corresponding multiples of the denominator.

Problem: You are planning a menu for a class party. Each student wiill receive one drink at the party. Your teacher tells you that, for a class of this size, you can expect 3 out of 5 students to prefer cola, whereas 2 out of 5 will prefer lemonade. If, in fact, this guideline proves true for your class of 30 students, how many students will want cola? How many will want lemonade?

Solution:

- Write ratios to compare the students who want cola and lemonade to the students in the class:

  3 colas for every 5 students; how many colas for 30 students?

  $$\frac{\text{3 colas}}{\text{5 students}} = \frac{\text{? colas}}{\text{30 students}}$$

  2 lemonades for every 5 students; how many lemonades for 30 students?

  $$\frac{\text{2 lemonades}}{\text{5 students}} = \frac{\text{? lemonades}}{\text{30 students}}$$

- Then find equivalent ratios for 30 students:

  $$\frac{\text{3 colas}}{\text{5 students}} = \frac{\text{3 colas}}{\text{5 students}} \times \frac{6}{6} = \frac{\text{3 colas} \times 6}{\text{5 students} \times 6} = \frac{\text{18 colas}}{\text{30 students}}$$

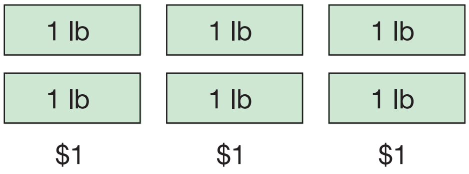
  $$\frac{\text{2 lemonades}}{\text{5 students}} = \frac{\text{2 lemonades}}{\text{5 students}} \times \frac{6}{6} = \frac{\text{2 lemonades} \times 6}{\text{5 students} \times 6} = \frac{\text{12 lemonades}}{\text{30 students}}$$

Fig. 3.14. Using equivalent ratios to solve the Class Party problem

Students should also have experiences solving problems involving rate. As the following classroom discussion shows, students benefit from focused problem-solving experiences that begin with problems that are intuitively simple to solve and progress to more complex problems. Solving simpler problems first will enable students to develop the understanding of the models and reasoning that are useful in solving more complex problems. During these experiences, teachers should help students transition from their own models to such models as strip diagrams.

Teacher: You buy 6 pounds of bananas for $3. How many pounds are you getting for $1?

Tom: I drew a diagram to show how to separate 6 pounds into 3 equal groups, one group for each dollar. There are 2 pounds in each group, so I am getting 2 pounds for $1.



Teacher: What operation are you using when you separate 6 into 3 equal groups?

Tom: Division.

Teacher: Let’s rearrange Tom’s drawing like this:

<table>
<tr><td>1lb</td><td>1lb</td><td>1lb</td><td>1lb</td><td>1lb</td><td>1lb</td></tr>
<tr><td colspan="2">$1</td><td colspan="2">$1</td><td colspan="2">$1</td></tr>
</table>

and then like this:

2 lb	2 lb	2 lb
$1	$1	$1

Teacher: Strip models like this one are powerful models because they can be used to model and solve many different types of problems. The model shows that you can buy 2 pounds of bananas for $1. It also reveals that this banana problem is a rate problem. Look back at the problem. Six pounds for 3 dollars is a rate that can be written as

$$\frac{6 \text{ pounds}}{3 \text{ dollars}}.$$

The equivalent ratio

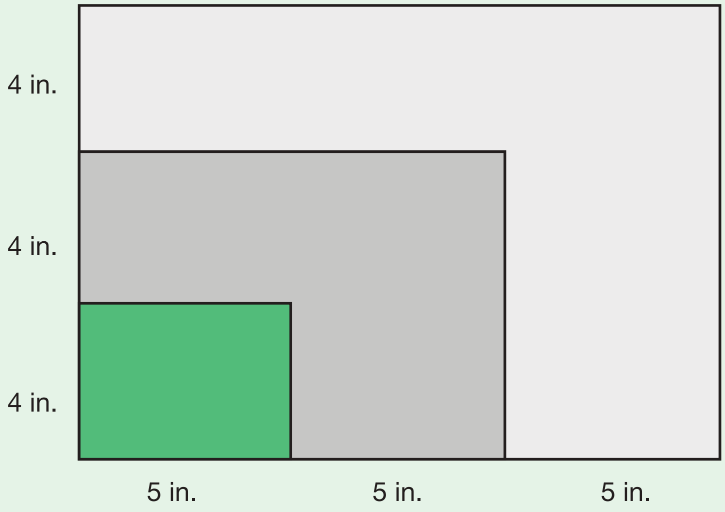
$$\frac{2 \text{ pounds}}{1 \text{ dollar}}$$

is a unit rate and is the solution to the problem. So the rate given in the problem can be written as a unit rate to find the solution.

After students use ratio and rate to analyze and solve simple problems such as the Banana Problem, teachers should present parallel problems in which the context or rate is changed. For example, present the Banana Problem context, but ask what would happen if you could buy 6 pounds for $2.40? Or 3 pounds for $6? Students should predict and then verify how the model would change, how the rate would change, how the answer would change, and so on. Presenting students with carefully crafted parallel problems that differ in specific ways helps teachers focus on different aspects of the concept. These same techniques can be used to explore different types of problems, for example, problems involving better buy, scale drawings, maps, and aspect ratio. Sample problems are shown in figure 3.15.

1. Aspect Ratio Problem:

For some rectangular TV screens, the aspect ratio is 4:5, which means that if one dimension is 4 units long, the other is 5 units long. What would be the length of a TV screen that has an aspect ratio of 4:5 whose width is 12 inches?



4:5 = 8:10 = 12:15
The length is 15 inches.

2. Lowest - Unit - Price Problem:

Callie's Cats sells cat food for \$0.89 a can. Pet Mart sells 5 cans of cat food for \$4.00. At Jay's Pets, cat food is 8 cans for \$6.50. At which store is cat food the least expensive? (Remember, fractions of cents are rounded up.)

Callie's Cats: $\frac{\$0.89}{1 \text{ can}}$

Pet Mart: $\frac{\$4.00}{5 \text{ cans}} = \frac{\$0.80}{1 \text{ can}}$

Jay's Pets: $\frac{\$6.50}{8 \text{ cans}} \approx \frac{\$0.82}{1 \text{ can}}$

Pet Mart has the least expensive cat food when compared by unit price.

Fig. 3.15. Sample problems involving ratio and rate

Strengthening Understanding of Ratio, Rate, and Unit Rate through Connections

As is apparent throughout this Focal Point, ratio and rate connect to various mathematical topics as well as contexts from other disciplines. For example, in science, social studies, humanities, and even sports, students encounter such rates as miles per hour (speed), meters per second (speed), pounds per square inch (pressure), grams per cubic centimeter (density), residents per square mile (population density), calories per gram (a measure in nutrition study), and hits per at-bat (batting average). Other connections occur in probability, measurement, and geometry.

Students connect ratios to probability when they write the probability of an event as a ratio of favorable outcomes to possible outcomes. For example, if students were to choose a marble randomly from the bag of marbles shown in figure 3.16, the probability of choosing a gray marble can be written as the ratio of the number of gray marbles to the number of marbles in all, or 2:7 or 2/7.

Students connect ratios to measurement when they learn how to convert from one measurement unit to another. For example, they can multiply by the ratio

$$\frac{12 \text{ inches}}{1 \text{ foot}}$$

to convert from feet to inches or multiply by the ratio

$$\frac{1 \text{ foot}}{12 \text{ inches}}$$

to convert from inches to feet. Students can also connect ratio to geometry and measurement when they explore how the changes in the dimensions of a figure affect its perimeter, area, or volume. The following example presents students with a problem in which they can use their understanding of ratios to deepen their understanding of relationships in geometric figures. Students are given the rectangles shown in figure 3.17.

After comparing the rectangles, students are asked to find the ratios of width to width

$$\left[\frac{2}{3}\right],$$

length to length

$$\left[\frac{4}{6} = \frac{2}{3}\right],$$

and area to area

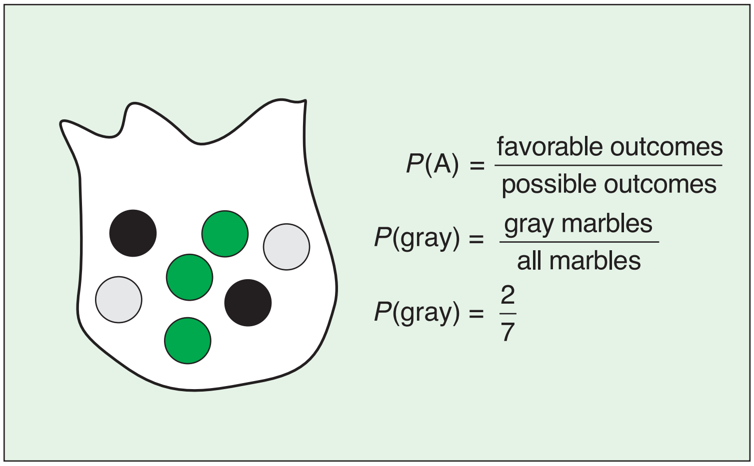


Fig. 3.16. Example showing how ratios connect to probability

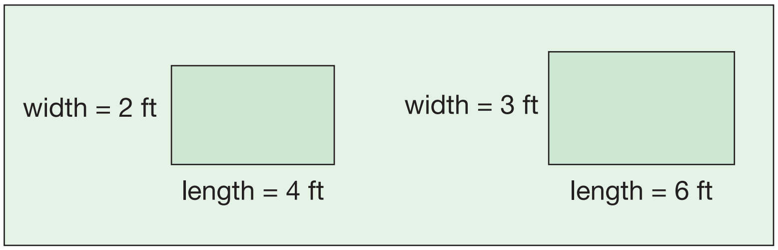


Fig. 3.17. Comparing rectangles

$$\left[\frac{8}{18} = \frac{4}{9}\right].$$

On the basis of their answers to questions like these in several different examples, students begin to conjecture that if *a/b* is the ratio of the corresponding linear dimensions of the two rectangles, then the ratio of the areas of the two rectangles will be $(a/b)^2$. They can apply the understanding of the area relationship to the volume relationship, as seen in the following example. Students are given the similar rectangular prisms shown in figure 3.18.

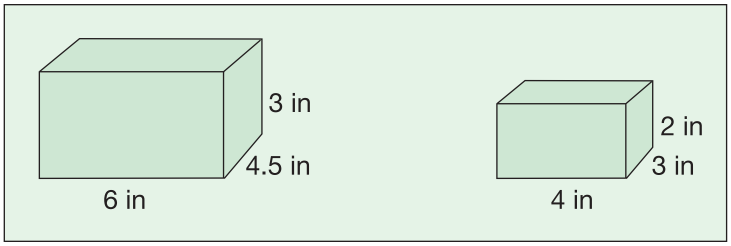


Fig. 3.18. Comparing rectangular prisms

After comparing the rectangular prisms, students are asked to find the ratios of corresponding dimensions

$$\left[\frac{4}{6} = \frac{3}{4.5} = \frac{2}{3}\right]$$

and the ratio of volumes

$$\left[\frac{24}{81} = \frac{8}{27}\right].$$

On the basis of their answers to these questions, and the understanding they developed with similar experiences with area, they begin to conjecture that if a/b is the ratio of the corresponding linear dimensions of the two rectangular prisms, then the ratio of the volumes of the two prisms will be $(a/b)^3$.

Connections in later grades

In grade 7 students will use their knowledge of ratio to develop an understanding of proportionality and use it to solve a variety of percent problems, including problems involving discounts, interest, taxes, tips, and percent increase and decrease. They will solve problems involving similar figures, both two-dimensional and three-dimensional. Students will graph direct variations, or direct proportions, using equations in the form $y = kx$. They will also identify the slope of a graph as the rate of change in the value of y compared to the value of x. They will distinguish direct variations from other relationships, including inverse variations.

In later grades students will recognize a direct variation, or proportional relationship, as shown in figure 3.19, as a special case of a linear function, as shown in figure 3.19 and 3.20.

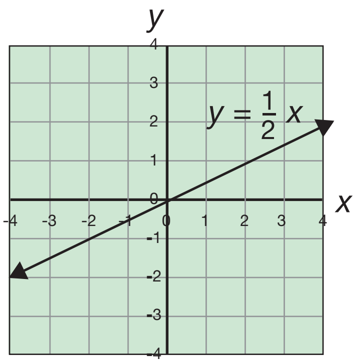


Fig. 3.19. Direct variation (proportional relationship) Constant of variation = Slope = $\frac{1}{2}$

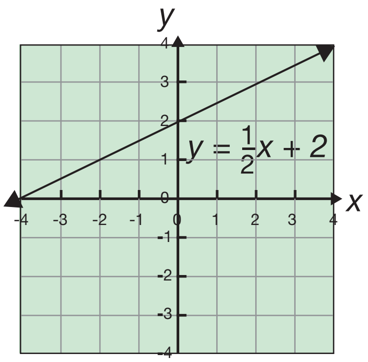


Fig. 3.20. Linear function (one that is not a proportional relationship) Slope = $\frac{1}{2}$

Students also will encounter π as the ratio of the circumference of any circle to its diameter. They will use scale factors (ratios) to solve problems involving enlargements and reductions. They will use trigonometric ratios to solve right-triangle problems.

Developing Depth of Understanding

Students have used various methods to solve problems involving comparison relationships before their formal introduction to ratio and rate. How can you construct problem-solving experiences that will help students connect their understanding of multiplication and division to the use of ratios and rate to solve these kinds of problems?

4 Focusing on Expressions and Equations

In grade 6, students develop an ability to write, interpret, and use mathematical expressions and equations. The focus is on understanding the similarities between numerical and algebraic expressions and equations. The eventual goal of this Focal Point is for students to develop the ability to use algebraic representations to generalize numerical relationships and express mathematical ideas concisely.

Instructional Progression for Expressions and Equations

The focus on expressions and equations in grade 6 is supported by a progression of related mathematical ideas before and after grade 6, as shown in table 4.1. To give perspective to the grade 6 work, we first discuss some of the important ideas that students focused on before grade 6 that prepare them for expressions and equations concepts in grade 6. At the end of the detailed discussion of this grade 6 Focal Point, we present examples of how students will use the expressions and equations understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books from NCTM, for example, *Focus in Grade 3*, *Focus in Grade 4*, *Focus in Grade 5*, *Focus in Grade 7*, and *Focus in Grade 8*.

Table 4.1 represents an instructional progression for the conceptual understanding of expressions and equations before, during, and after grade 6.

Early Foundations in Mathematical Expressions and Equations

Before entering the grade 6 classroom, students are expected to learn many concepts and skills that they can use to understand their work in expressions and equations. Students add, subtract, multiply, and divide whole numbers. They use words and numbers to describe patterns and number relationships. They also use formulas to represent and determine the perimeter and area of simple polygons and the volume of simple polyhedra. All these skills work together to facilitate students' understanding of the concepts in this Focal Point.

In students' previous work with whole-number computation, they develop an understanding of numerical expressions and equations. For example, when they evaluate 45 × 21, they are evaluating a numerical expression. When they write their answer as 45 × 21 = 945, they are writing a numerical equation and beginning to understand that the "=" symbol is used to show the equality between two expressions. Students will use their knowledge of

Table 4.1

Grade 6: Focusing on Expressions and Equations—Instructional Progression for Expressions and Equations

Before Grade 6	Grade 6	After Grade 6
Students develop an understanding of, and fluency with, addition, subtraction, multiplication, and division of whole numbers. Students use words and numbers to describe patterns and relationships, for example, "The number of legs is 4 times the number of chairs." Students use formulas to represent and determine areas of simple polygons.	Students begin to develop the ability to generalize numerical relationships and express mathematical ideas concisely using expressions and equations (e.g., three more as $x + 3$, doubling as $2n$, commutativity as $a + b = b + a$). Students use the properties of equality (e.g., equals added to equals are equal) to create equivalent equations. Students solve simple one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation and understand that the solutions of an equation are the values of the variables that make the equation true. Students develop the skills to use expressions, equations, and formulas to solve problems (e.g., applying the understanding of equivalent expressions to decide between the use of $5x$ or $2x + 3x$ in the solution to a problem).	Students develop a thorough understanding of negative numbers (including negative decimals and decimal fractions). Students use models of negative numbers to represent and justify rules for adding, subtracting, multiplying, and dividing with negative numbers. Students develop efficient, accurate, and generalizable methods for operating with negative numbers. Students use linear equations in one variable and rational numbers to solve word problems. Students translate among algebraic, geometric (graphical), numerical (tabular), and verbal representations of linear functions. Students analyze and solve problems using linear equations and systems of linear equations.

whole-number computation to write, interpret, and use both numerical and algebraic expressions and equations.

In previous grades, students study patterns. They use a variety of methods to describe those patterns. For example, to analyze the pattern of number of chairs to number of legs, students might make the chart shown in figure 4.1.

Chairs	1	2	3	4
Legs	4	8	12	16

Fig. 4.1. Chart that shows the pattern of number of chairs to number of legs

If they observe that they can multiply the number of chairs by 4 to get the number of legs, they might describe the pattern as "The number of legs is 4 times the number of chairs." They might also write the relationship as a numerical expression to find the number of legs for 5 chairs as $5 \times 4 = 20$. These understandings form the foundation that enables students to learn in this Focal Point how to generalize a rule for finding the number of legs for any number of chairs by writing an algebraic expression $4c$, where c is the number of chairs.

Previously, students also used formulas to represent and determine the perimeter and area of simple polygons. For example, in grade 4 students define the area of a two-dimensional figure to be the number of square units needed to cover the figure. They apply this definition to rectangles whose sides were integral lengths and their understanding of array models to discover they can multiply the number of rows of square units by the number of columns of square units to find the total number of squares covering the rectangle. This foundation leads them to hypothesize that, for any rectangle, the area would equal the product of the length times the width of the rectangle. As they are introduced to the formula $A = l \times w$, they begin to use algebraic thinking for problem types that previously had been solved with arithmetic thinking. In this Focal Point, students will use their familiarity with geometry formulas to deepen their understanding of algebraic representations.

Focusing on Writing, Interpreting, and Using Expressions and Equations

The ultimate goal of this Focal Point is for students to learn to generalize numerical relationships and express mathematical ideas concisely using expressions and equations. For example, they should realize that *three more than a number* can be written $x + 3$, doubling can be written $2n$, and the relationship between a and b in the table in figure 4.2 can be represented by the equation $b = 4a$.

a	1	2	3	4
b	4	8	12	16

Fig. 4.2. Table showing the relationship $b = 4a$

To fully realize the goal of this Focal Point, students will need to be able to write, interpret, and use *algebraic* expressions and equations. We distinguish this situation from a purely arithmetic approach in that we use symbols (typically letters) to represent numbers in an expression or equation. These symbols can be used in different ways, as the following examples demonstrate.

- $n + 4 = 10$: n is not explicitly stated in the equation, but it has only one value, $n = 6$, that results in an equality.
- $y = x + 4$: x and y are pairs of numbers that satisfy the relationship represented by the equation.
- $y = mx + b$: In this slope-intercept form of the equation for a straight

> line, m and b are unspecified constants (the slope and the y-coordinate of the point where the line crosses the y-axis), whereas x and y are pairs of numbers that satisfy the relationship once m and b are specified.

It is common to refer to n, x, y in the examples above as *unknowns* or *variables,* but few people would call m and b variables. Because the term *variable* is in wide use in curriculum materials, the term is used in this publication. Experience and mathematical judgment are the keys to avoiding misusing the term.

However, before students can work meaningfully with algebraic representations and use algebraic representations to generalize mathematical relationships, it is essential for them to first explore the numerical representations with which they are familiar. Although this Focal Point addresses skills used in evaluating algebraic expressions and solving algebraic equations, the learning of these skills is not the main goal for students in prealgebra grades. Rather, the purpose of this Focal Point in grade 6 is for students to acquire a sense of the relationship between the numerical and the algebraic. Activities, problems, and discussions should work together to build for grade 6 students a belief that algebraic expressions behave in the same way as numerical expressions. For example, once students understand properties applied to numerical expressions and equations, they can more fully understand these same properties applied to algebraic expressions and equations, as shown by the examples of the subtraction property of equality and the commutative property of addition in figure 4.3.

<table>
<tr><th colspan="2">Subtraction Property of Equality</th></tr>
<tr><td>$4 + 6 = 10$
$4 + 6 - 6 = 10 - 6$
$4 = 4$</td><td>$n + 6 = 10$
$n + 6 - 6 = 10 - 6$
$n = 4$</td></tr>
<tr><th colspan="2">Commutative Property of Addition</th></tr>
<tr><td>$5 + 2 = 2 + 5$</td><td>$a + b = b + a$</td></tr>
</table>

Fig. 4.3. Numerical and algebraic examples of properties

By connecting representations of numerical relationships that students have acquired prior to grade 6 to the more formal algebraic representations, students can build a strong understanding of how the algebraic representation is related to what they know, the numerical relationship. Once students have formed this solid foundation, they will be able to use algebraic symbols flexibly to represent mathematical relationships and to solve problems.

One important reason for students to learn about algebraic representations is to use expressions and equations to model and solve problems. To attain this goal, teachers should be purposeful in the progression of problems students need to solve. The initial problems should include the expression or equation and ask the student to use the algebraic representation to solve the problem and then explain why the answer solves the problem. As students progress in their understanding, teachers should expect students to both write and solve their own algebraic representations. More about this problem-solving progression is provided in the Strengthening Understanding of Expressions and Equations through Problem Solving section of this Focal Point.

Reflect As You Read

Before continuing, think about the unique role language plays in mathematics. When do students experience difficulties? Why does this happen? What teacher-moves do you take or strategies do you use to make the issue of mathematical language easier?

Using language to develop understanding of expressions and equations

The study of expressions and equations comes with its own set of characteristic language. As students develop the ability to understand and use this language, their understanding of algebraic representation will deepen. Some important vocabulary associated with this Focal Point includes *numerical expressions*, *numerical equations*, *algebraic expressions*, *algebraic equations*, *evaluate*, *solve*, *variable*, and *constant*. A numerical expression consists of symbols and numbers, for example 2 + 3,

$$\frac{(10-2)^2}{4},$$

and 453. Expressions do not include an equals sign (=). A numerical equation is a mathematical sentence that uses the equals sign to indicate that two expressions have the same value, for example 4 + 5 = 9,

$$\frac{(10-2)^2}{4}=17-1,$$

and 45 = 45. In numerical expressions and equations, every term is a number whose value is known. Students have represented numerical expressions and equations in a variety of ways, as shown in the examples in figure 4.4.

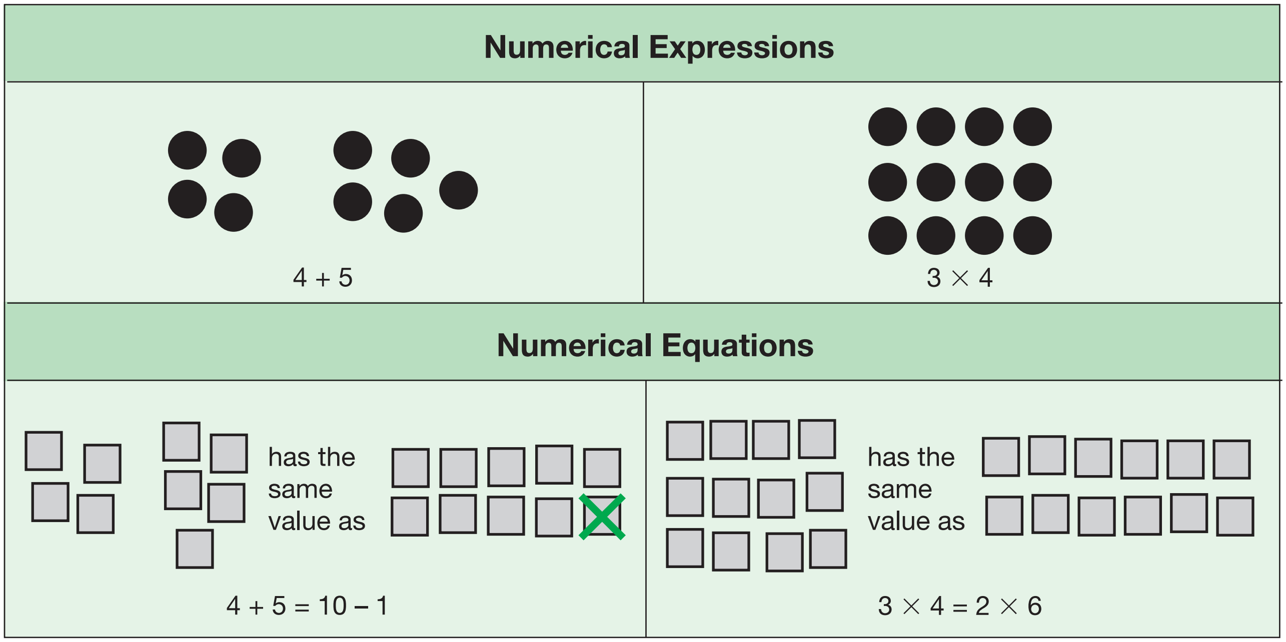


Fig. 4.4. Ways to represent numerical expressions and equations with counters and square tiles

Expressions and equations can also be algebraic. Algebraic expressions and equations consist of at least one unknown value, represented by a variable. A variable is a symbol, such as a letter, that represents a number. The following are examples of algebraic expressions and equations: x, $5c$, $(y + 10)^2$, $2n = 6$, $3(x + 1)^2 = 300$, $x = 1$, $7 + x = 12 - y$. As shown in figure 4.4, students have used a variety of models to represent numerical expressions and equations, including counters and square tiles. In grade 6, they should be guided to more generalizable representations using algebra tiles or other algebra models. Students can use algebra tiles to represent both algebraic and numerical expressions and equations. Although algebra tile sets vary, it is customary to use a small square such as □ for 1 (since its area is 1 unit by 1 unit, or 1 square unit) and a rectangle such as ▭ for the variable x (since it can be described as having dimensions of x units by 1 unit, with an area of x square units). Figure 4.5 shows how students can use algebra tiles to model algebraic equations and expressions.

As students begin to explore algebraic representation, they need to understand that a variable represents any value that makes the statement true. A variable, then, may represent one value, many values, or no value. For example, in the equation $n + 2 = n + 3$, no value for n makes the equation true, because adding 2 to a number cannot give the same result as adding 3 to the same number. In the equation $n + 2 = 5$, n has one value that makes the equation true, and that value is 3. In the equation $a + b = b + a$, a and b have infinitely many pairs of values that make the equation true.

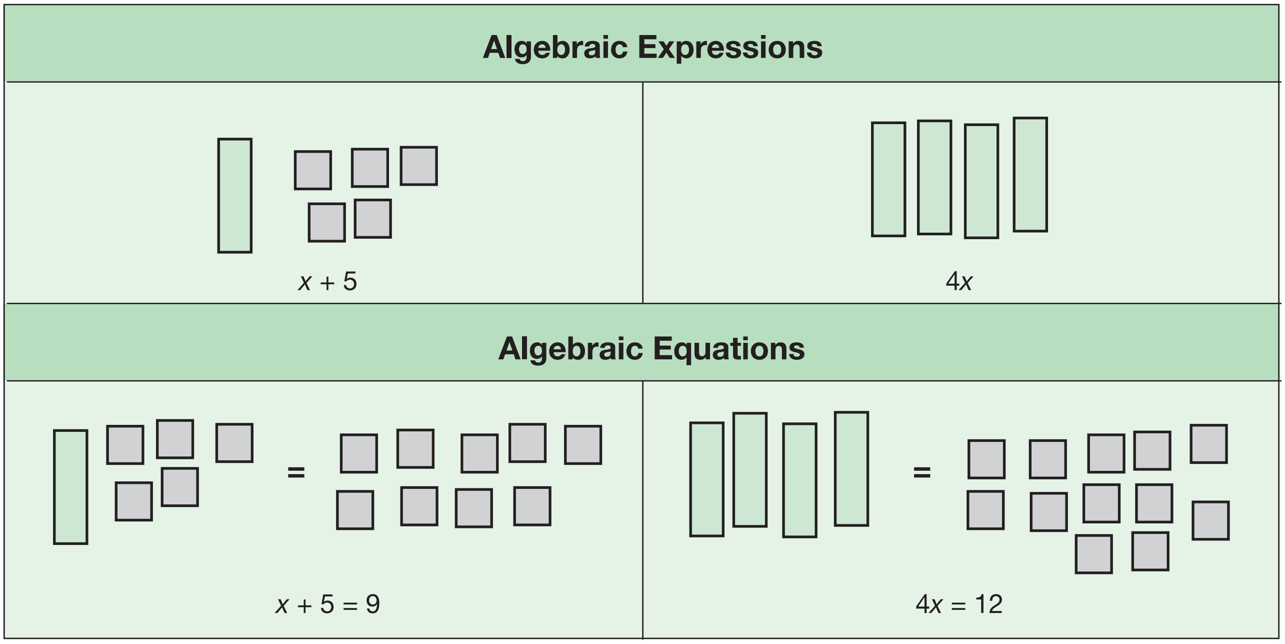


Fig. 4.5. Ways to represent algebraic expressions and equations with algebra tiles

Reflect As You Read

As you consider the importance of the equals sign as presented in the following text, try this short assessment to discover whether your students truly understand the meaning of the equals sign as a symbol that means "is the same as" or whether they think of it as a symbol that means "to calculate."

On an index card, choose the correct answer to fill in the blank.

6 + 8 = ______+ 5

a) 9 *b*) 14 *c*) 19 *d*) 14 and 19

An important aspect of studying equations is applying the appropriate meaning for the equals sign. The equals sign in an equation sets up a condition in which the expression on the left of the equals sign has the same value as the expression on the right. This condition determines which numbers solve the equation. Students can use the idea of a balance to help develop their understanding of an equation as a set of conditions. Representing equations with the balance model builds on students' understanding that if the sides of the balance are level, they have the same "weight." Students can translate this analogy to modeling equations. If they represent one expression on one side of the balance and another expression on the other side of the balance and the sides are the same height, the first expression has the same value as the other expression and the equation is true. If the two sides of the balance are level, then the fulcrum of the balance becomes a representation of the equals sign. One of the balances in figure 4.6 shows that 2 × 5 = 10.

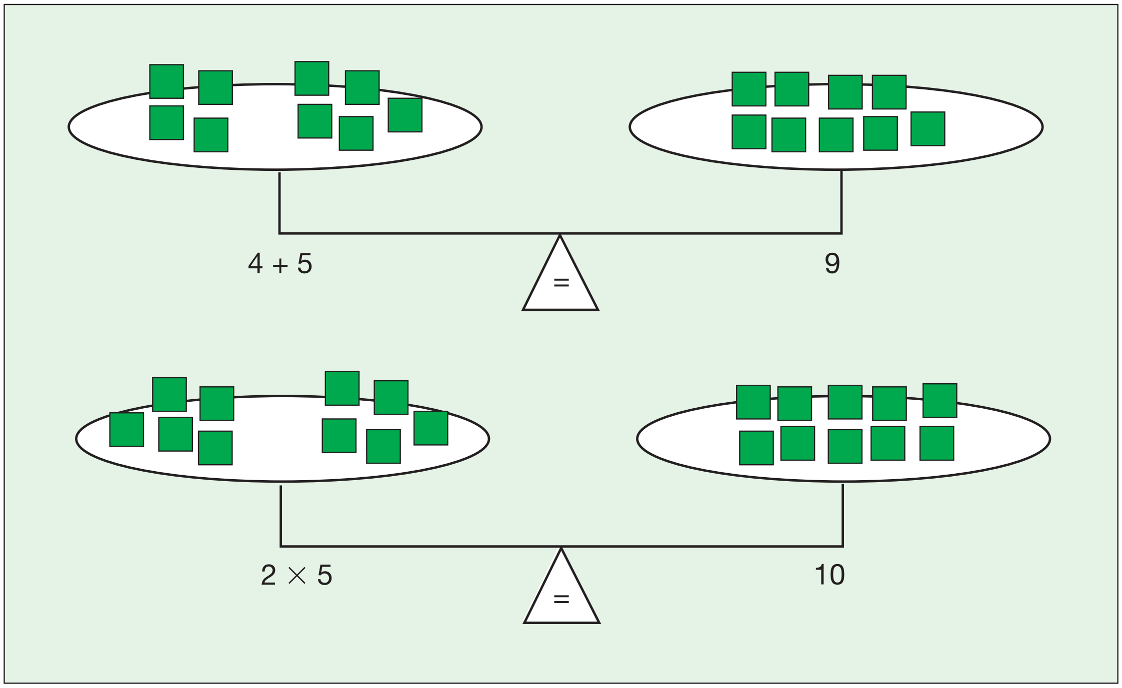


Fig. 4.6. Using a balance to model equations

Students should also be made aware that any equation can be rewritten by switching the left and right sides. For example, $2 \times 5 = 10$ can be written $10 = 2 \times 5$. The property stating this fact formally is the *symmetric property of equality* and can be represented algebraically as if $a = b$, then $b = a$ for any numbers a and b. As part of a curriculum focused on expressions and equations, students will have opportunities to *evaluate* and *solve*. When students evaluate, they find the value of an expression. Previous to grade 6, students have used their computational skills to evaluate numerical expressions. For example, they have evaluated 5×7 by multiplying 5 by 7 to get 35. In grade 6, students will learn how to evaluate algebraic expressions and equations. For example, they might evaluate $5n$ for $n = 7$ to get 35, or they might solve $5n = 35$ to get an answer of $n = 7$. Figure 4.7 summarizes the connections among the vocabulary associated with expressions and equations.

Developing expressions and equations to represent a sequence

As students study expressions and equations, teachers need to help students move from representations that are often numerical, to more powerful and generalizable representations, to formal algebraic representations. The following classroom discussion illustrates how this transition might be accomplished.

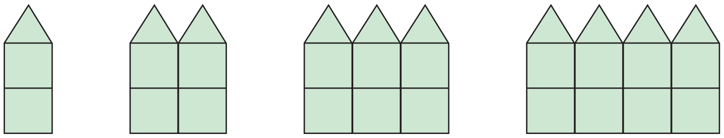
Teacher: Here is a representation of houses that we will call the 1st, 2nd, 3rd, and 4th house. Let's assume that the pattern of adding one more column

Operation	Type of Expression	Numerical Expression	Algebraic Expression	Numerical Equation	Algebraic Equation
Addition	sum	$5 + 3$	$n + 3$	$5 + 3 = 8$	$n + 3 = 8$
Subtraction	difference	$5 - 3$	$n - 3$	$5 - 3 = 2$	$n - 3 = 2$
Multiplication	product	3×5	$3 \times n$	$3 \times 5 = 15$	$3 \times n = 15$
Division	quotient	$5 \div 3$ or $\frac{5}{3}$	$n \div 3$ or $\frac{n}{3}$	$15 \div 3 = 5$ or $\frac{15}{3} = 5$	$n \div 3 = 5$ or $\frac{n}{3} = 5$

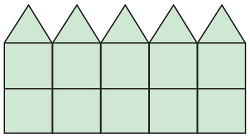
Fig. 4.7. Table summarizing the vocabulary associated with expressions and equations

to the previous house continues. If the pattern continues, how many total blocks are in the 5th house?

Rita: I understood how to solve this problem. I could see that every house had 2 squares with a triangle on top added to the house before it. So I drew the 5th house. Then I counted the shapes. There were 15 shapes in the 5th house.



Tavon: I see how Rita solved the problem, but I realized that I didn't have to actually draw the houses. I could just write down the number of shapes in each house. There were 3 shapes in the 1st house, 6 in the 2nd, 9 in the 3rd, and so on. So I made a table that shows how the number of the house, 1, 2, 3, 4, goes with the number of shapes for that house, 3, 6, 9, 12. Then I used the "add 3" pattern to determine that the 5th house will have 15 shapes.

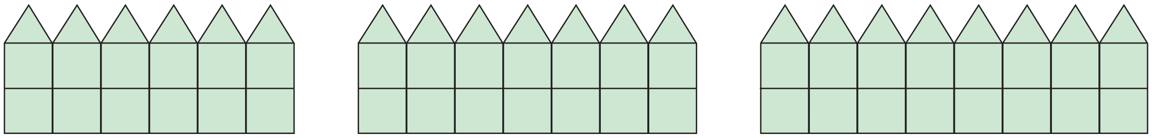


Teacher: Tavon used "12 + 3" to find the answer. This is called a *numerical expression*. It is a numerical expression because it contains numbers and an operation symbol. There are no unknown values in a numerical expression; all the values are specified. As Tavon said, the value of 12 + 3 is 15, so we can write 12 + 3 = 15. This is called a *numerical equation* because it is a mathematical sentence formed by placing an equals sign between the expression 12 + 3 and the number 15. The equals sign indicates that the value of one side of the equation is the same as the value on the other side. Since we know that 12 + 3 has the same value as 15, we know that this equation is true. As with numerical expressions, there are no unknowns in a numerical equation. All the values are specified.

Teacher: Rita, can you use your method to find the number of shapes in the 8th house?

Rita: Yes, I draw the 6th, 7th, and 8th houses and then count the shapes in the 8th house. There are 24 shapes in the 8th house.

Teacher: I understand. Tavon, can you show and explain how to find the number of shapes in the 8th house?



Tavon: I already know there are 15 shapes in the 5th house and that the rule for the pattern is add 3. So I write the number of shapes in the 5th house and then the 6th, 7th, and 8th houses by adding 3 each time. So in the 8th house there are 24 shapes.

5th	6th	7th	8th
15	18	21	24

Teacher: I see. Let's think about using these two strategies to find the number of shapes in the 100th or 500th house.

At this point in the discussion, the teacher can help to guide students to see the limitations of Rita's and Tavon's representations. Rita's strategy is a counting strategy. She has to build each model and count. Tavon's strategy is a *recursive* strategy because he uses a number in the sequence to get the next number in the sequence. His representation works only to find the number of shapes in a house if the number of shapes in the house before is known. For example, finding the number of shapes in the 100th house with this representation requires knowing the number of shapes in the 99th house. Discussions like these give teachers the opportunity to guide students to a representation that is more generalizable, and then, eventually, to an algebraic representation that can be used to find the number of shapes for any given house number, as is shown in the continuation of the class discussion.

Teacher: Tavon found a pattern in the number of shapes in each house. We have learned to use tables to organize patterns, so let's organize Tavon's pattern in a table. We can use one row to show the number of the house. We can use the other row to show the number of shapes in that house.

Teachers should guide students to make the following table.

House Number	1	2	3	4	5	6	7	8
Number of Shapes	3	6	9	12	15	18	21	24

Teachers should also encourage students to create the table in a vertical format. They will use this format in later grades when they begin to graph relationships.

House Number	Number of Shapes
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24

The table is a representation that enables many students to more easily identify and then generalize the relationship between the house number and the number of shapes in the house, that is, that the number of shapes is 3 times the house number. Once students understand this representation, the teacher can then guide them to use an algebraic expression and equation connected to this pattern to solve the problem.

Teacher: The table helps us identify relationships. What is the relationship between each house number and the number of shapes in that house?

Tavon: I see it now. The number of shapes in a house is 3 times the house number. The first house has 1 × 3 = 3, or 3 shapes; the second house has 2 × 3 = 6, or 6 shapes, the 8th house has 8 × 3 = 24, or 24 shapes, and so on. I think I see how to use this relationship to find the 500th house!

House Number	Number of Shapes (3 Times House Number)
1	3
2	6
3	9
4	12
500	?

Teacher: Tavon, can you explain it to us?

Tavon: Yes. I can multiply 500 by 3 to get the number of shapes in the 500th house: 500 × 3 = 1,500, so there are 1,500 shapes in the 500th house.

Tavon is using algebraic thinking when he says, "The number of shapes in a house is 3 times the house number." A table like the one shown in figure 4.8 can help students like Tavon connect their algebraic reasoning to numerical representations.

House Number	Number of Shapes (3 Times House Number)
1	1 × 3 = 3
2	2 × 3 = 6
3	3 × 3 = 9
4	4 × 3 = 12
500	500 × 3 = 1,500

Fig. 4.8. Table that shows numerical equations used to solve the House Problem

This representation helps students gain a deeper understanding of the problem and leads them closer to the algebraic representation because they can see the generalized relationship of "the number of shapes in a house is equal to the house number times 3." The strategy of multiplying the house number by 3 to get the number of shapes is an explicit strategy. Both explicit and recursive approaches are quite valuable, and students should have experiences with both. When an easy-to-express explicit formula exists, it is certainly desirable to find that formula.

As is clear from the following continued classroom discussion, students who understand the representations up to this point are ready to move to the formal algebraic representation.

Teacher: Now that we know that we can multiply the house number to find the number of shapes in any house, it would be useful to have a way to write this relationship concisely. That is one purpose of algebraic representations, to write generalized mathematical relationships concisely, or efficiently. To write the relationship between the house number and the number of shapes in the house, we need to replace the numbers that can change with some other symbol. We can choose any symbols to represent the numbers, but we usually use letters. Cara, can you use a sentence to describe the relationship between the house number and the number of shapes?

Cara: The house number times 3 is equal to the number of shapes.

Teacher: So, let's consider Cara's statement, "The house number times 3 is equal to the number of shapes." If we let the letter h represent the house number and the letter s represent the number of shapes, we can write $s = 3 \times h$ or $s = 3h$. This equation is a very concise mathematical sentence that says the same thing as what Cara said. And we can use this equation to determine the number of shapes that would correspond to any house. For example, if the house number is 150, the corresponding number of shapes is $s = 3 \times 150 = 450$. Let's add another row to our table to express this relationship.

The teacher can use a diagram like the one in figure 4.9 to help students understand how to write the relationship as an algebraic equation.

Fig. 4.9. Diagram to help students write an equation to represent the

$$\underbrace{\text{The house number}}_{h}\ \underbrace{\text{times 3}}_{\times 3}\ \underbrace{\text{is equal to}}_{=}\ \underbrace{\text{the number of shapes.}}_{s}$$

or

$$3h = s$$

$$s = 3h$$

House Problem

Teachers should then have students look back at a table like the one in figure 4.8.

House Number	Number of Shapes (3 Times House Number)
1	$1 \times 3 = 3$
2	$2 \times 3 = 6$
3	$3 \times 3 = 9$
4	$4 \times 3 = 12$
500	$500 \times 3 = 1{,}500$
h	$h \times 3 = s$ *or* $3h = s$

Teacher: Now we have a powerful representation of the problem that we can use to find the number of shapes in any house number. Rita, can you explain how we can use the equation to find the number of shapes in the 700th house?

Rita: Yes, I can. The house number is 700, so $h = 700$. I can put 700 in for h in the equation, so $700 \times 3 = s$. I multiply 700 by 3 and get 2,100. If the house number is 1,700, $h = 1{,}700$ and $1{,}700 \times 3 = 5{,}100$. Wow, I can do this for any number, even 2,944,234,115,393. This is a lot faster than drawing houses.

Teacher: That's right. Algebraic representations help us write mathematical relationships in efficient ways so that we can use them to solve problems.

During the discussion, the teacher should explain that all the equations in figure 4.10 represent the same relationship and can be used to solve the House Problem.

$$3 \times h = s$$
$$h \times 3 = s$$
$$3h = s$$
$$s = 3h$$

Fig. 4.10. Different ways to write the same equation

After students write the equation to find the number of shapes given the number of houses, teachers can point out that the same equation can be used to find the house number given the number of shapes, as is evidenced in the continuation of the classroom discussion.

Teacher: You know you can use the equation $s = 3h$ to find s, the number of shapes in a house, given h, the house number. But consider the following problem: A house has 36 shapes. What house number is it? Does anyone know how we can use the same equation to find the solution to this problem?

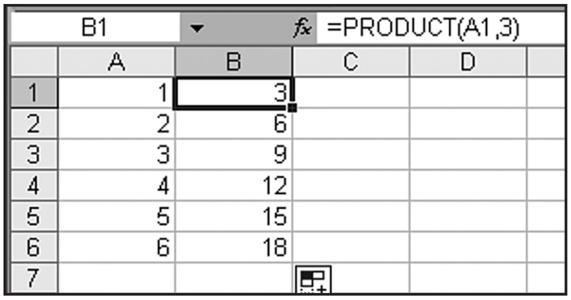
Ben: I think I do. I know the number of shapes: $s = 36$. So I can put in 36 for s in the equation.

$$s = 3h$$
$$36 = 3h$$

Then I need to find a number that when multiplied by 3 gives 36. I know that 12×3 is 36, so h must be 12.

At some time during or after the discussion of a problem such as the house problem, it would be beneficial for the teacher to point out that algebraic representation is an essential element in the use of technology. For example, figure 4.11 shows how a spreadsheet program might be used after students have observed that the number of shapes is 3 times the house number. Column A shows the house numbers. Column B shows the number of shapes in each house. To use column A to generate column B, an equation that multiplies the number in A by 3 needs to be written. The understanding of algebraic representation fosters the understanding that students need to be able to construct that formula.

As students begin to learn how to find the values for variables that solve equations, they need to develop an understanding of the properties of equality for addition, subtraction, multiplication, and division. For example, the



B1 =PRODUCT(A1,3)

	A	B	C	D
1	1	3		
2	2	6		
3	3	9		
4	4	12		
5	5	15		
6	6	18		
7				

Fig. 4.11. A spreadsheet program worksheet showing a "3 times" relationship using properties of equality

addition property of equality states that if you add a number to both sides of a true equation, the equation stays true. Teachers should present these properties first using numerical equations. Students' understanding of how the properties are applied in numerical equations will lead to the belief that the properties work in the same way in algebraic equations. Examples such as those found in figure 4.12 will be helpful in accomplishing this goal.

Addition Property of Equality
If $a = b$, then $a + c = b + c$.

Examples:

$$25 - 10 = 15$$
$$25 - 10 + 10 = 15 + 10$$
$$25 = 25$$

$$25 - 10 = 15$$
$$25 - 10 + 3 = 15 + 3$$
$$18 = 18$$

Subtraction Property of Equality
If $a = b$, then $a - c = b - c$.

Examples:

$$2 + 4 = 6$$
$$2 + 4 - 4 = 6 - 4$$
$$2 = 2$$

$$2 + 4 = 6$$
$$2 + 4 - 1 = 6 - 1$$
$$5 = 5$$

Multplication Property of Equality
If $a = b$, then $a \times c = b \times c$, for $c \neq 0$

Examples:

$$\frac{20}{5} = 4$$
$$5 \times \frac{20}{5} = 5 \times 4$$
$$20 = 20$$

$$\frac{20}{5} = 4$$
$$8 \times \frac{20}{5} = 8 \times 4$$
$$32 = 32$$

Division Property of Equality
If $a = b$, then $a \div c = b \div c$, for $c \neq 0$.

Examples:

$$6 \times 8 = 48$$
$$\frac{6 \times 8}{6} = \frac{48}{6}$$
$$8 = 8$$

$$6 \times 8 = 48$$
$$\frac{6 \times 8}{2} = \frac{48}{2}$$
$$24 = 24$$

Fig. 4.12. Examples of the properties of equality

Teachers should give students opportunities to apply the properties of equality to numerical expressions until students begin to understand that the properties of equality are true for any number in any equation, other than dividing by 0 since that is undefined.

It is also beneficial for teachers to revisit the balance model as they discuss the properties of equality with students. For example, the balances in figure 4.13 show how to model the addition property of equality.

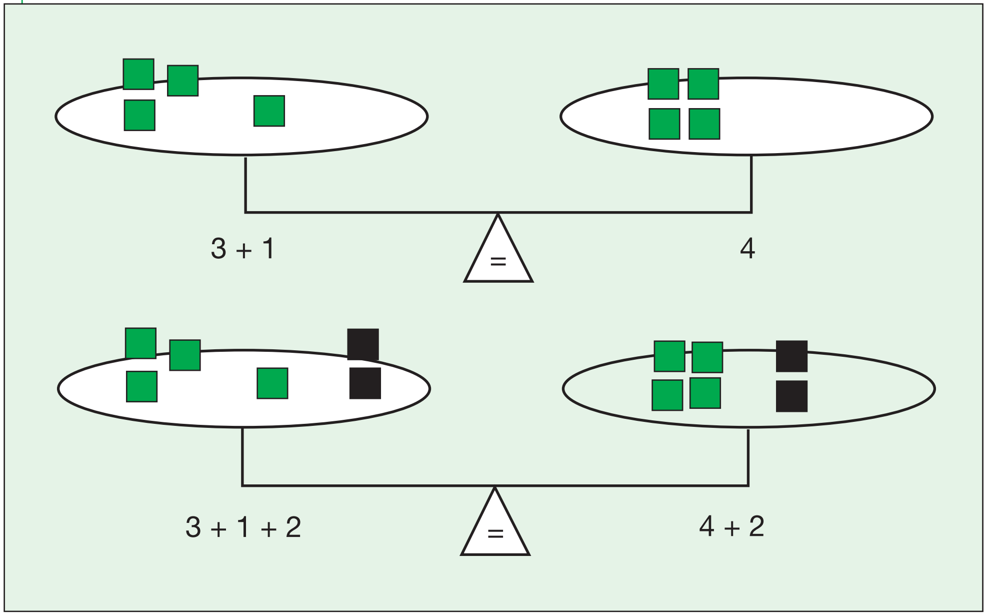


Fig. 4.13. Balance models showing the addition property of equality

Once students have gained an understanding of the equality properties in numerical equations, they can move to applying the properties to algebraic equations. A balance model can be used to demonstrate the multiplication property of equality when an unknown is involved, as shown in figure 4.14.

Once students have built the understanding of the properties of equality, they are ready to learn how to use these properties to solve equations. It is essential that students gain this understanding first so that they can apply the properties of equality to algebraic equations in a meaningful way. Teachers can revisit previously solved problems to help students understand how to use the equality properties to solve equations, as is evident in the following classroom discussion.

Teacher: Let’s look at the equation that we used to represent the House Problem, $s = 3h$. Remember that we used the equation to solve the problem: “A house has 36 shapes. What house number is it?” We substituted 36 for s in the equation $s = 3h$ to get the equation $36 = 3h$. Then we thought, “36 is equal

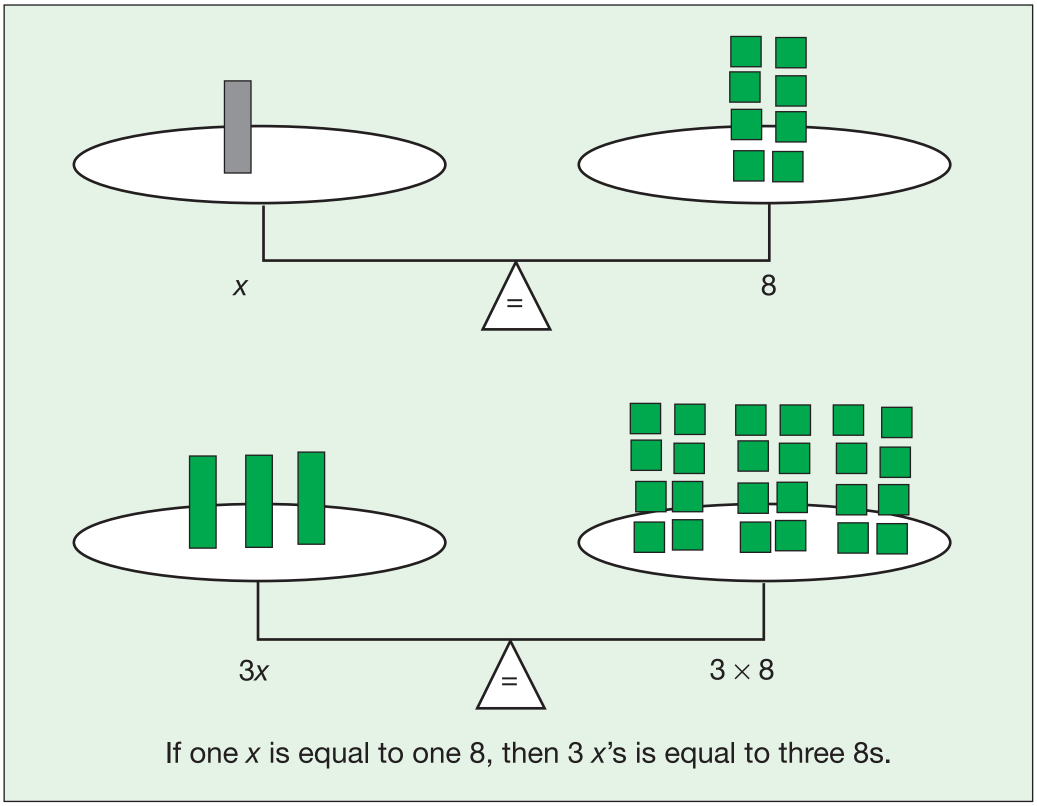


Fig. 4.14. Balance models showing the multiplication property of equality in an algebraic equation

to 3 times what number?" We know that 3 times 12 is 36, so house number 12 has 36 shapes. If you know your multiplication facts, it was pretty easy to find *h*. But what would happen if we wanted to solve the same problem for a house with 201 shapes? You might not immediately know what number multiplied by 3 results in the product 201. If you know the properties of equality, you can use them to help you solve this problem. What equation would you write to find the house number that has 201 shapes?

Tara: We know *s* and are trying to find *h*, so $201 = 3h$.

Teacher: The properties of equality can help us find the answer. We have 3 *h*'s and we want to know 1 *h*. When you have 3 of something and you want to know the value of 1 of that thing, you divide. Or, you can think, since the *h* is multiplied by 3, we need to use the inverse operation, division, to "undo" the multiplication and divide $3h$ by 3 to have $1h$, or just *h*. To keep the equation true, we must use the division property of equality and divide the other side of the equation by 3. Remember, when $3h = 36$, we knew that $h = 12$ because $3h \div 3 = 36 \div 3$, or $1h = 12$. We can apply the same property of equality in this situation.

$$201 = 3h$$

$$\frac{201}{3} = \frac{3h}{3}$$

$$\frac{201}{3} = 1h$$

Now we can see that to find the value of h, we need to divide 201 by 3 to get 67. So when $s = 201$, $h = 67$. How does this solve our problem?

Marc: Since h is equal to 67 and h is the house number, then house 67 has 201 shapes. And if I put 67 in place of h in the equation, then $3 \times 67 = 201$ is a true statement.

Teachers can use examples like this one and the ones shown in figure 4.15 along with concrete representations, such as algebra tiles and balances, to teach students how to use inverse operations and the respective equality property to solve different types of equations. In each example, it is important to discuss how the choice of the operation and number used was made to find the value of the variable.

$x - 10 = 15$ $x - 10 + 10 = 15 + 10$ $x = 25$	Add 10 to both sides of the equation because 10 is subtracted from x and addition and subtraction are inverse operations.
$x + 4 = 6$ $x + 4 - 4 = 6 - 4$ $x = 2$	Subtract 4 from both sides of the equation because 4 is added to x and addition and subtraction are inverse operations.
$\frac{x}{5} = 4$ $5 \bullet \frac{x}{5} = 5 \bullet 4$ $x = 20$	Multiply both sides of the equation by 5 because x is divided by 5 and multiplication and division are inverse operations.
$6x = 48$ $\frac{6x}{6} = \frac{48}{6}$ $x = 8$	Divide both sides of the equation by 6 because x is multiplied by 6 and multiplication and division are inverse operations.

Fig. 4.15. Examples of using the equality properties to solve equations

Strengthening Understanding of Expressions and Equations through Problem Solving

Teachers should construct problem-solving experiences with a purposeful progression as they focus students' attention on how to write, interpret, and use equations and expressions. In initial problems, students are given both the problem and the expression or equation that can be used to solve the problem. The action on the part of the student in this initial phase, then, is to correctly use the algebraic representation to solve the problem and then explain why the answer solves the problem. In this phase of the problem-solving progression, students should encounter uses of both expressions and equations to solve problems. The emphases during this phase are for students to begin, through example, to understand how the relationships in problems can be represented by algebraic representations and for students to gain an understanding of why a number is or is not a solution to the problem. To this end, it is beneficial for teachers to look at correct solutions to problems but also to have students examine solutions that are not correct, and explain why not.

In the second phase of a focused problem-solving approach, students should begin to write their own algebraic representations for the relationships in the problem and then use that representation to find the solution or solutions. Again, these experiences should include problems in which students write expressions and ones in which they write equations. The examples in figure 4.16 illustrate this progression.

Problem 1: Jane and Stephen took photos on a field trip. Jane took 2 times as many photos as Stephen. Stephen took 6 photos. How many photos did Jane take? Use the expression $2s$, where s stands for the number of photos Stephen took, to solve the problem.

Solution: Stephen took 6 photos, so substitute 6 for s.

$$2s$$
$$2(6)$$
$$12$$

Jane took 12 photos. This makes sense because Jane took 2 times as many photos as Stephen and 12 is 2 times 6.

Problem 2: Jane and Stephen took photos on a field trip. Jane took 2 times as many photos as Stephen. Jane took 6 photos. How many photos did Stephen take? Use the expression $2s = j$, where s stands for the number of photos Stephen took and j stands for the number of photos Jane took, to solve the problem.

Solution: Jane took 6 photos, so substitute 6 for j.

$$2s = j$$

$$2s = 6$$

Use the division property of equality to find the value for s.

Fig. 4.16. Example problems that show problem-solving progressions

$$\begin{gathered} 2s = 6 \\ \frac{2s}{2} = \frac{6}{2} \\ s = 3 \end{gathered}$$

Stephen took 3 photos. This makes sense because Jane took twice as many photos as Stephen, and 6 is 2 times 3. Also, if I substitute $s = 3$ in the equation, the equation is still true:

$$\begin{gathered} 2s = 6 \\ 2(3) = 6 \end{gathered}$$

Problem 3 Arthur and Don are reading books. Don read 4 times as many pages as Arthur. Write an expression that represents the number of pages Don read. Use your expression to find the number of pages Don read if Arthur read 16 pages.

Possible Solution Expression: $4a$, where a is the number of pages Arthur read. Substitute 16 for a:

$$\begin{gathered} 4a \\ 4(16) \\ 64 \end{gathered}$$

If Arthur read 16 pages, Don read 64 pages. This makes sense because Don read 4 times as many as Arthur, and 4 times 16 is 64.

Problem 4 Arthur and Don are reading books. Don read 4 times as many pages as Arthur. Write an equation that represents the number of pages Arthur read if Don read 400 pages.

Possible Solution Equation: $d = 4a$, where d is the number of pages Don read and a is the number of pages Arthur read.

$$\begin{aligned} d &= 4a \\ 400 &= 4a \end{aligned}$$

Use the division property of equality to find the value of a.

$$\begin{aligned} d &= 4a \\ 400 &= 4a \\ \frac{400}{4} &= \frac{4a}{4} \\ 100 &= a \end{aligned}$$

Arthur read 100 pages. This makes sense because Don read 4 times as many pages as Arthur, and 400 is 100 times 4. Also, if I substitute $a = 100$ in the equation, the equation is still true.

$$\begin{aligned} 400 &= 4a \\ 400 &= 4(100) \end{aligned}$$

Fig. 4.16. Example problems that show problem-solving progressions—*Continued*

Note that in problems 2 and 4, students can also use the division property of equality before they substitute to write an equivalent equation in terms of *s* or *a*, as shown in figure 4.17. Teachers should carefully construct problems so that students begin to see these connections and understand that relationships can be represented algebraically in different ways.

Students should also use their understanding of expressions and equations to solve problems involving geometric figures and the formulas for area, perimeter, and volume. One such problem is described in figure 4.18.

$$2s = j$$
$$\frac{2s}{2} = \frac{j}{2}$$
$$s = \frac{j}{2}$$
$$s = \frac{6}{2}$$
$$s = 3$$

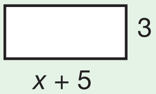
Problem 2

$$d = 4a$$
$$\frac{d}{4} = \frac{4a}{4}$$
$$\frac{d}{4} = a$$
$$\frac{400}{4} = a$$
$$100 = a$$

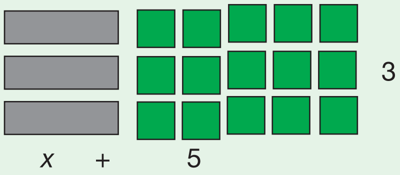
Problem 4

Fig. 4.17. Using the division property of equality to write equivalent expressions before substitution

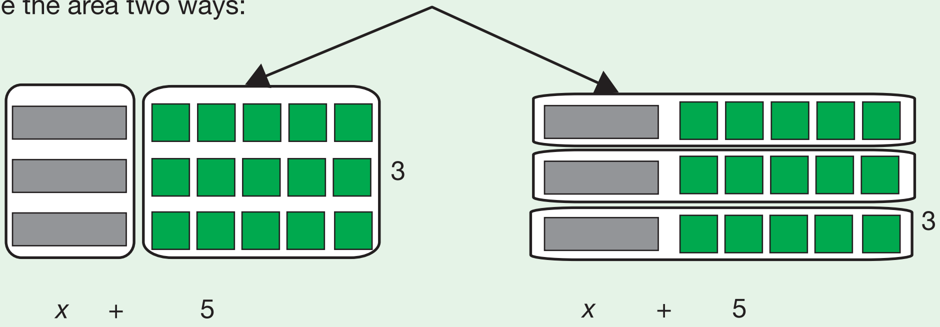
Problem 1: Write two equivalent expressions that represent the area of this rectangle.



Solution:
Use algebra tiles to represent the rectangle:



Write the area two ways:



$3x + (3 \cdot 5) = 3x + 15$

$3(x + 5)$

Fig. 4.18. Problem involving geometric figures and measurement formulas

Problem 1: The table shows how feet and inches are related.

Feet x	Inches y
1	12
2	24
3	36
4	48

Write an equation that shows the relationship between feet and inches and has y alone on one side. Use your equation to find the number of inches in 15 feet. Then write an equivalent equation that has x alone on one side. Use your new equation to find the number of feet in 300 inches.

Solution:

$$y = 12x$$
$$y = 12(15)$$
$$y = 80$$

15 feet = 80 inches

$$y = 12x$$
$$\frac{y}{12} = \frac{12x}{12}$$
$$\frac{y}{12} = x$$
$$\frac{300}{12} = x$$
$$25 = x$$

300 inches = 25 feet

Problem 2: The table shows how pounds and ounces are related.

Pounds x	Ounces y
1	16
2	32
3	48
4	64

Write an equation that shows the relationship between ounces and pounds and has y alone on one side. Use your equation to find the number of ounces in $9\frac{1}{2}$ pounds. Write an equivalent equation that has x alone on one side. Use your new equation to find the number of pounds in 100 ounces.

$$y = 16x$$
$$y = 16\left(9\frac{1}{2}\right)$$
$$y = 152$$

$9\frac{1}{2}$ pounds = 152 ounces

$$y = 16x$$
$$\frac{y}{16} = \frac{16x}{16x}$$
$$\frac{y}{16} = x$$
$$\frac{100}{16} = x$$
$$6\frac{1}{4} = x$$

100 ounces = $6\frac{1}{4}$ pounds

Fig. 4.19. Problems involving converting units of measurement

As another application of equations and expressions in grade 6, students should solve problems involving measurement conversions. The problem in figure 4.19 shows examples of these kinds of problems.

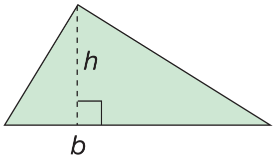
Strengthening Understanding of Expressions and Equations through Connections

As stated at the beginning of this Focal Point section, the most important goal for students is to acquire an intuitive belief that algebraic expressions behave in the same way as numerical expressions. Having acquired this belief, they can use expressions and equations to generalize properties they recognize as they work with numbers. As summarized in the table in figure 4.20, given each property, teachers can have students verify numerical examples of a property and generalize the property by stating it algebraically. Only one numerical example is shown in each instance, but students should verify as many numerical examples as is necessary for students to begin to understand that the property applies to all numbers. When students represent algebraically what they already know numerically, they are practicing the language of algebra.

Name of Property	Numerical Example	Algebraic Representation
Commutative property of addition	$5 + 6 = 6 + 5$	$a + b = b + a$
Commutative property of multiplication	$5 \times 2 = 2 \times 5$	$ab = ba$
Associative property of addition	$(2 + 3) + 6 = 2 + (3 + 6)$	$(a + b) + c = a + (b + c)$
Associative property of multiplication	$(4 \times 3) \times 2 = 4 \times (3 \times 2)$	$(ab)c = a(bc)$
Identity property for addition	$9 + 0 = 0 + 9 = 9$	$a + 0 = 0 + a = a$
Identity property for multiplication	$1 \times 10 = 10 \times 1 = 10$	$1a = a \times 1 = a$
Property of zero for multiplication	$0 \times 7 = 0$	$0a = 0$
Distributive property	$6(2 + 5) = 6(2) + 6(5)$ $5(8 - 2) = 5(8) - 5(2)$	$a(b + c) = ab + ac$ $a(b - c) = ab - ac$

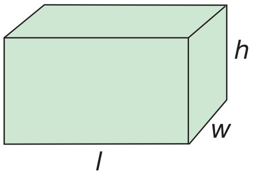
Fig. 4.20. Connecting algebraic representations to properties

As demonstrated throughout this Focal Point section, algebraic representation can be used to generalize numerical relationships. Moreover, algebraic representation can be connected to every topic in mathematics that can be generalized. Any time a variable is used to represent a possible value, an algebraic representation is being used. And because variables are used to represent numbers in all areas of mathematics, algebraic representation is connected to all areas of mathematics. For example, in two-dimensional geometry, A, b, and h represent area, base, and height in the formula for the area of a triangle—



$$A = \frac{1}{2}bh.$$

The formula represents how the area, base, and height of a triangle are related.



In three-dimensional geometry, V, l, w, and h represent volume, length, width, and height in the formula for the volume of a rectangular prism—$V = lwh$. The formula represents how the volume, length, width, and height of a rectangular prism are related.

Connections in later grades

In grade 7 students will extend their study of area and volume as they study circles, prisms, and cylinders—decomposing these figures into component parts. In doing so, they will use algebraic representations as they develop formulas. For example, they will learn that the volume of a cylinder and the volume of a rectangular prism can both be described by the same formula—$V = Bh$, where B is the area of the base, h is the height, and Bh is the algebraic representation of the product of the area of the base and the height. In grade 7 students will work with rational numbers, including positive and negative integers, fractions, and decimals, using them to simplify algebraic expressions, evaluate algebraic expressions, and solve algebraic equations. For example, they will solve such equations as $2.5x = 10$. In later grades, students will use their understanding of algebraic representations as they translate among algebraic, geometric (pictorial and graphical), numerical (tabular), and verbal representations of linear functions and use linear equations and systems of linear equations to analyze and solve problems, as illustrated in the example in figure 4.21.

Verbal

Problem: Al has a bicycle rental business. He charges an initial fee of \$2 plus \$3 per hour for each rental. Show this information in a table, an equation, and a graph. For the graph, assume that Al is willing to rent bicycles for parts of an hour.

Tabular

Al's Bicycle Rentals	
Number of hours, x	Cost, *y*
1	\$ 5
2	\$ 8
3	\$11
4	\$14

Algebraic: $y = 3x + 2$

Graphical

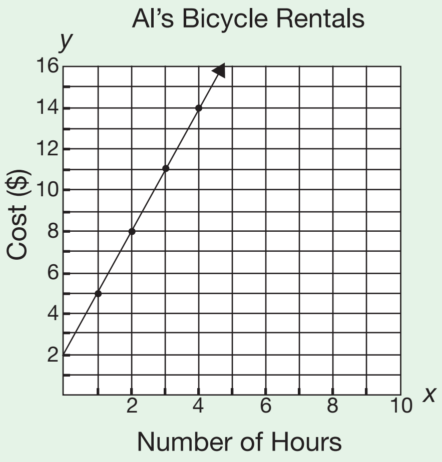


Fig. 4.21. Example showing the connection between algebraic representations and linear functions

Students will also connect algebraic representation to their future study of probability. Students will learn, for example, that $P(A)$ represents the probability of event A and $P(B)$ represents the probability of event B. If events A and B are complementary, then the sum of their probabilities is 1. That relationship is represented by the equation $P(A) + P(B) = 1$.

Developing Depth of Understanding

After reading about focusing on equations and expressions as suggested in the grade 6 Algebra Focal Point, consider what activities you do or plan to do to help students gain the understanding that algebraic representations work in the same way as numerical representations. How can you move students from their understanding of the numerical to an understanding of the algebraic?

References

Donovan, M. Suzanne, and John D. Bransford, eds. *How Students Learn: Mathematics in the Classroom.* Washington, D.C.: National Research Council, 2005.

Fuson, Karen C., and Aki Murata. "Integrating the NRC Principles and the NCTM Process Standards: Cognitively Guided Teaching to Individualize Instruction within Whole-Class Activities and Move All Students within Their Learning Path." *National Council of Supervisors of Mathematics Journal* 10 (Spring 2007): 72–91.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. *Adding It Up: Helping Children Learn Mathematics.* Washington, D.C.: National Research Council, 2001.

Lampert, Magdalene. "Choosing and Using Mathematical Tools in Classroom Discourse." In *Advances in Research on Teaching,* vol. 1, edited by Jere Brophy, pp. 223–64. Greenwich, Conn: JAI Press, 1989.

Mack, Nancy K. "Learning Fractions with Understanding: Building on Informal Knowledge." *Journal for Research in Mathematics Education* 21 (January 1990): 16–32.

Mirra, Amy. *Focus in Grades 3–5.* Reston, Va.: National Council of Teachers of Mathematics, 2008.

———. *Focus in Grades 6–8.* Reston, Va.: National Council of Teachers of Mathematics, 2009.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

_____. *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

_____. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.* Reston, Va.: NCTM, 2006.

_____. *Focus in Grade 3.* Reston, Va.: NCTM, 2009a.

_____. *Focus in Grade 4.* Reston, Va.: NCTM, 2009b.

_____. *Focus in Grade 5.* Reston, Va.: NCTM, 2009c.

_____. *Focus in Grade 7.* Reston, Va.: NCTM, 2010a.

_____. *Focus in Grade 8.* Reston, Va.: NCTM, 2010b.

Webb, David C., Nina Boswinkel, and Truus Dekker. "Beneath the Tip of the Iceberg: Using Representations to Support Student Understanding." *Mathematics Teaching in the Middle School* 14 (September 2008): 110–13.

Focus in Grade 6

Teaching with Curriculum Focal Points

Focus in Grade 6 shows teachers and other educators how they can incorporate NCTM's Curriculum Focal Points for Pre-K–8 into their current mathematics curricula. The book provides practical ideas for bringing focus to mathematics learning and instruction in the classroom. Intended as a professional development tool, it presents self- and group-reflection tasks, sample student work, and other tools that teacher educators can use with the preservice teachers in their classes.

By focusing more intensely on fewer topics at each grade level, students gain a deeper understanding of mathematical ideas. This volume will help teachers think about what a focused curriculum means and how they might begin to build focus into their existing curricula.